interested to find out the time it will take to complete this project.

What jobs are critical to the completion of the project in time, etc ?

Solution. For this, it is necessary to find out the earliest and the latest completion time for each activity in the network. The earliest and the latest times are re-calculated by using *'forward pass'* and *'backward pass'* computations, respectively.

To understand the procedure, we define :

$E_i =$ the earliest expected occurrence time of event i,

$L_j =$ the latest allowable event occurrence time for event j.
This is the latest time by which the event j must be started without increasing the project duration.

$D_{ij} =$ the expected duration to complete the activity i–j.

The solution now starts by the forward pass computation.

Step 1. ***Determination of Earliest Time (E_j) : Forward Pass Computation***

The purpose of the forward pass computation is to find out earliest start times for all the activities. For this, it is necessary to assign some initial value to the starting node 1. Usually this value is taken to be zero so that the subsequent earliest time could be interpreted as the project duration up to that point in question.

Rules for the computation are as follows :

***Rule* 1.** Initial event is supposed to occur at time equal to zero, that is, $E_1 = 0$.

***Rule* 2.** Any activity can start immediately when all preceding activities are completed.
The earliest time E_j for node j is given by $E_j = \max_i \, [E_i + D_{ij}]$,

where i is the collection of nodes which precede node j.

***Rule* 3.** Repeat **step 2** for the next eligible activity until the end node is reached.

Numerical Calculation :

Consider the network (*Fig.* 25.20.) by assumption $E_1 = 0$ and $E_2 = \max_i \, [E_i + D_{i2}]$.

For node 2, node 1 is the only predecessor and hence $i = 1$ contains only one element. Therefore,

$$E_2 = E_1 + D_{12} = 0 + 2 = 2.$$

Likewise, values of E_3, E_4, E_5 and E_6 can be computed as :

$E_3 = E_1 + D_{13} = 0 + 2 = 2$, $E_4 = E_1 + D_{14} = 0 + 2 = 2$, $E_5 = E_2 + D_{25} = 2 + 4 = 6$, $E_6 = E_3 + D_{36} = 2 + 5 = 7$.

Consider node 7, where there are three emerging activities, *i.e.* $E_7 = \max_i \, (E_i + D_{i7})$,

The collection i consists of nodes 3,4 and 6 that are preceding node 7. Therefore,

$$E_7 = \max \, [\, E_3 + D_{37} = 2 + 8 = 10, E_4 + D_{47} = 2 + 4 = 6, E_6 + D_{67} = 7 + 0 = 7\,] = 10$$

$$E_8 = \max \, [E_5 + D_{58} = 6 + 2 = 8, E_6 + D_{68} = 7 + 4 = 11] = 11$$

$$E_9 = \max \, [E_8 + D_{89} = 11 + 3 = 14, E_7 + D_{79} = 10 + 5 = \mathbf{15^*}] = 15$$

and $$E_{10} = E_9 + D_{9,10} = 15 + 4 = 19.$$

From this computation, it can be inferred that this job will take 19 days to finish as this is the longest path of the network. Activities along this longest path are : 1—3, 3—7, 7—9 and 9—10. This longest path is called the critical path. In any network, it is not possible that there can be only one critical path. For example, if in the above network, let $D_{36} = 6$ days, then two critical paths exist having the same duration for completion of project.

Step 2. ***Determination of Latest Time (L_i) : Backward Pass Computation***

In forward pass computation, the earliest time when a particular activity will be completed is known. It is also seen that some activities are not critical to the completion of the job. The question, a manager would like to ask is : Can their starting time be delayed so that the total completion time is still the same ? Such a question may arise while scheduling the resources such as : manpower, equipment, finance and so on. If delay is allowable, then what can be the maximum delay ? For, this is the latest time for various activities which is desired. The backward pass computation procedure is used to calculate the latest time for various activities. In the forward pass computation, assignment of $E_1 = 0$ was arbitrary, likewise for the backward pass computation, it is possible to assign the project terminal event the date on which the project should be over. If no such date is prescribed, then the covention is of setting latest allowable time determined in forward pass computation.

Rules of the backward pass computation are as follows :

Rule 1. Set $L_i = E_i$ or T_S

where T_s is the scheduled date for completion and E_i is the earliest terminal time.

Rule 2. $L_i = \min\limits_j \{L_j—D_{ij}\}$, *i.e.* the latest time for activities is the minimum of the latest time of all succeeding activities reducing their activity time.

Rule 3. Repeat *rule* 2 until starting activity is reached.

Latest times for activities of the network are calculated below :

By *rule 1*, set $L_{10} = 19$. Applying *rule 2*, it is to determine L_9, L_8 and L_7,

$$L_9 = \min_j \{L_j - D_{9,j}\} = 19 - 4 = 15 \text{ for } j = 10.$$

$$L_8 = \min_j \{L_j - D_{8,j}\} = L_9 - D_{8,9} = 15 - 3 = 12 \qquad (j \text{ contains only one node 9})$$

$$L_7 = \min_j \{L_j - D_{7,j}\} = L_9 - D_{7,9} = 15 - 5 = 10 \qquad (j \text{ contains node 9}).$$

Now consider node 6. for this node, there are two succeeding activities, namely 6—8 and 6—7.

Hence, $$L_6 = \min_{j=(7,8)} [L_j - D_{6,j}] = \min \begin{bmatrix} L_7 - D_{6,7} \\ L_8 - D_{6,8} \end{bmatrix} = \min \begin{bmatrix} 10 - 0 = 10 \\ 12 - 4 = 8^* \end{bmatrix} = 8$$

Similarly, $$L_5 = L_8 - D_{5,8} = 12 - 2 = 10, \qquad L_4 = L_7 - D_{4,7} = 10 - 4 = 6$$

$$L_3 = \min \begin{bmatrix} L_6 - D_{3,6} \\ L_7 - D_{3,7} \end{bmatrix} = \min \begin{bmatrix} 8 - 5 = 3 \\ 10 - 8 = 2* \end{bmatrix} = 2, L_2 = L_5 - D_{2,5} = 10 - 4 = 6$$

$$L_1 = \min_{j=(2,3,4)} [L_j - D_{1,j}] = \min \begin{bmatrix} 6 - 2 = 4 \\ 2 - 2 = 0* \\ 6 - 2 = 4 \end{bmatrix} = 0.$$

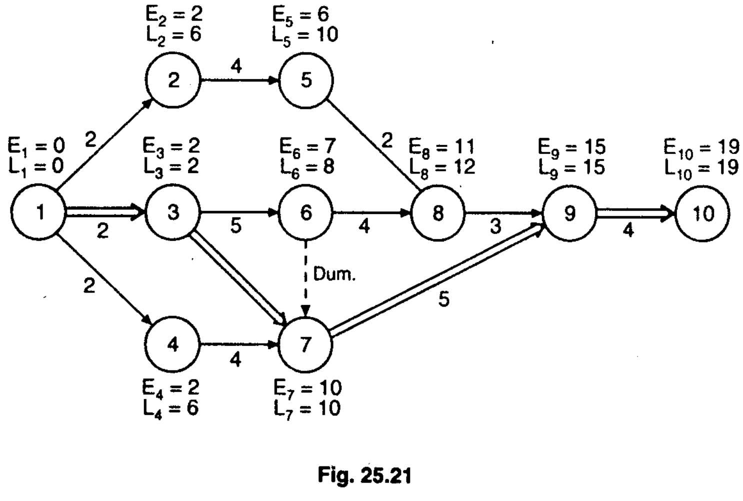


Fig. 25.21

The minimum value of $L_1 = 0$ is no surprising result. Since, started with $L_i = E_i$, it is always possible to have $L_1 = 0$. If this is not so, it means that some error is made in calculations of forward pass or backward pass values. *Fig.* 25.21 shows earliest and latest times of each event.

Recall that path 1—3—7—9—10 was defined as the critical path of this network. Along this path, it is observed that the latest and earliest times are the same implying that any activity along this path cannot be delayed without affecting the duration of the projcet.

Step 3. *Computation of Floats.*

By definition, for the activity 8—9, the float is one day ($L_8 - E_8 = 12 - 11 = 1$). This float represents the amount by which this particular activity can be delayed without influencing the duration of the project.

Also, by definition, free float, if any, will exist only on the activities merge points. To illustrate the concept of free float, consider path 1—2—5—8—9, total float on activity 8—9 is one day and since this is the last activity prior to merging two activities, this float is *free float* also. Similarly, consider the activity 5—8 which has a total float of 4 days but has only 3 days of free float because 1 day of free float is due to the activity 8—9. If the activity 5—8 is delayed up to three days, the early start time of no activity in the network will be affected. Therefore, the concept of free float clearly states that the use of free float time will not influence any succeeding activity float time.

If free float of any activity comes out to be negative, it is taken zero.

For example, independent float of (1, 2) = free float of (1, 2) − $(L_1 - E_1) = 0 - (0 - 0) = 0$.

Step 4. *To Identify Critical Path*

The earlier calculation shows that the path or paths which have *zero float* are called the *critical* ones. If this logic is extended little further, it would provide a guide rule to determine the next most critical path, and so on. Such an information will be useful for managers in the control of projects. In this example, path 1—3—8—9—10 happens to be next to critical path; because it has float of one day on many of its activities.

Table 25.2.

Activity $(i-j)$	Duration (D_{ij})	Start		Finish		Float		
		Earliest	Latest	Earliest	Latest	Total	Free	Independent
(1)	(2)	(3) E_i	(4) = (6) − (2)	(5) = (3) + (2)	(6) L_j	(7) = (4) − (3)	(8) = $E_j - E_i - D_{ij}$	(9) = (8) − $(L_i - E_i)$
1-2	2	0	4	2	6	4	0	0
1-3	2	0	0	2	2	0	0	0
1-4	2	0	4	2	6	4	0	0
2-5	4	2	6	6	10	4	0	0
3-6	5	2	3	7	8	1	0	0
3-7	8	2	2	10	10	0	0	0
4-7	4	2	6	6	10	4	4	0
5-8	2	6	10	8	12	4	3	0
6-8	4	7	8	11	12	1	0	0
7-9	5	10	10	15	15	0	0	0
8-9	3	11	12	14	15	1	1	0
9-10	4	15	15	19	19	0	0	0

The method discussed earlier is easily adoptable on computers. In the case of small networks, we can perform most of the calculations right on the diagram. In an event that a person would like to use tableau format to find floats, etc, such methods are also available. Table 25.2 summarizes float times and other information.

Q. 1. Define the following terms with reference to a PERT chart : (i) Total float, (ii) Free float, (iii) Independent float.

2. The local chapter of an institute is planning a dinner meeting with a nationally known speaker and you are responsible for organising it. How could PERT/CPM methodology be useful for this simulation ? **[CA. (Nov) 92]**

3. What is a project ? Give two examples. List the important four district features that are common to all projects. **[CA (May) 93]**

4. Define a dummy arrow used in a network. State two purposes for which it is used. Mention four conventions that are used in drowings the network. **[CA (Nov) 91]**

Example 2. *A project consists of a series or tasks labelled A, B,..., H, I with the following relationships (W < X, Y, means X and Y cannot start until W is completed; X, Y < W means W cannot start until both X and Y are completed). With this notation, construct the network diagram having the following constraints :*

$$A < D, E\ ;\ B, D < F\ ;\ C < G\ ;\ B < H;\ F, G < I.$$

Find also the optimum time of completion of the project, when the time (in days) of completion of each task is as follows :

Task	:	A	B	C	D	E	F	G	H	I
Time	:	23	8	20	16	24	18	19	4	10

Solution. The resulting network is shown in *Fig.* 25.22. The dummy activities D_1 and D_2 are introduced to establish the correct precedence relationship. The nodes of the project are numbered so that their ascending order shows the direction of progress in the project :

To determine the critical path (*optimum time of completion of the project*), compute the earliest start E_i and latest finish L_j for each task (i, j). The calculations are done as follows :

$$E_1 = 0,\ E_2 = E_1 + D_{12} = 0 + 20 = 20,\ E_3 = E_1 + D_{13} = 0 + 23 = 23.$$

To determine the value of E_4, since there are two incoming tasks (1, 4) and (3, 4),

$$E_4 = \max_{i=1,3} [E_i + D_{i4}] = \max [E_1 + D_{14}, E_3 + D_{34}] = \max [0 + 8, 23 + 16] = 39.$$

This procedure is repeated to compute all E_j. Thus

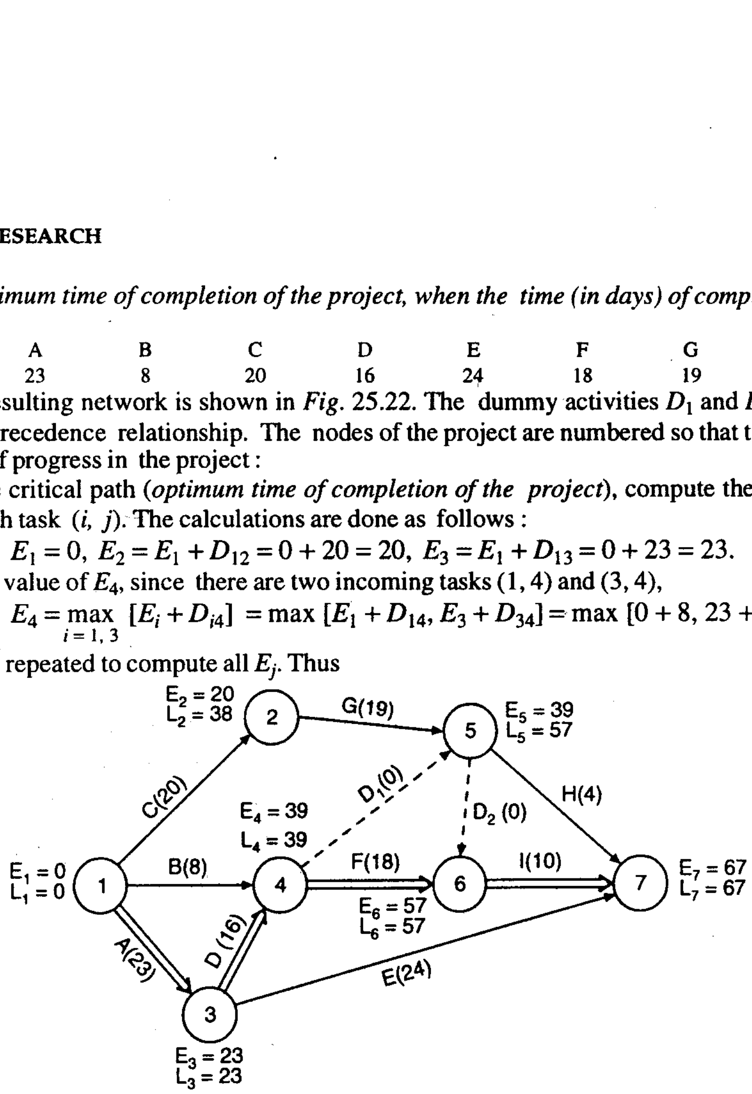


Fig. 25.22

$$E_5 = \max_{i=2,4} [E_i + D_{i5}] = \max [39 + 0, 20 + 19] = 39,\ E_6 = \max_{i=4,5} [E_i + D_{i6}] = \max [39 + 18, 39 + 0] = 57$$

$$E_7 = \max_{i=3,5,6} [E_i + D_{i7}] = \max [23 + 24, 39 + 4, 57 + 10] = 67.$$

The value of L_i are calculated proceeding backwards as follows :

$$L_7 = E_7 = 67, L_6 = L_7 - D_{67} = 67 - 10 = 57$$

$$L_5 = \min_{j=6,7} [L_j - D_{5j}] = \min [57 - 0, 67 - 4] = 57$$

$$L_4 = \min_{j=5,6} [L_j - D_{4j}] = \min [57 - 0, 57 - 18] = 39$$

$$L_3 = \min_{j=4,7} [L_j - d_{3j}] = \min [39 - 16, 67 - 24] = 23$$

$$L_2 = L_5 - D_{25} = 57 - 19 = 38$$

$$L_1 = \min_{j=2,3,4} [L_j - D_{1j}] = \min [38 - 20, 23 - 23, 39 - 8] = 0.$$

Table 25.3. Network Analysis Table

Task (i, j)	Normal Time (D_{ij})	Earliest Time		Latest Time		Float Time $(L_j - D_{ij}) - E_i$
		Start (E_i)	Finish $(E_i + D_{ij})$	Start $(L_j - D_{ij})$	Finish (L_j)	
(1, 2)	20	0	20	18	38	18
(1, 3)	23	0	23	0	23	0
(1, 4)	8	0	8	31	39	31
(2, 5)	19	20	39	38	57	18
(3, 4)	16	23	39	23	39	0
(3, 7)	24	23	47	43	67	20
(4, 5)	0	39	39	57	57	18

Continued →

(4, 6)	18	39	57	39	57	0
(5, 6)	0	39	39	57	57	18
(5, 7)	4	39	43	63	67	24
(6, 7)	10	57	67	57	67	0

From this table, critical nodes are for the tasks (1, 3) → (3, 4) → (4, 6) → (6, 7). Thus critical path is 1 → 3 → 4 → 6 → 7 as shown by double line arrows () in the figure. This path represents the longest possible duration to complete the entire project.

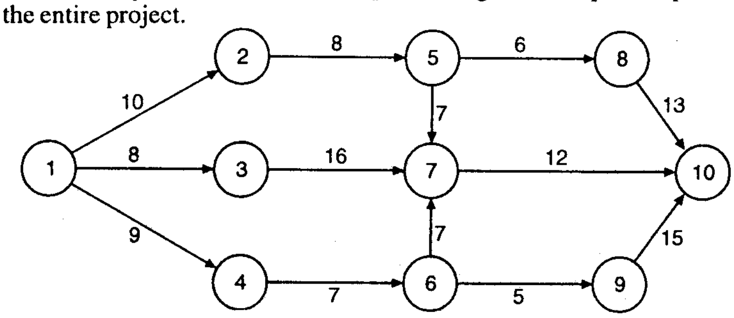


Fig. 25.23.

Example 3. *Determine early start (T_E) and late start (T_L) in respect of all node points and identify critical path in respect of the following network.*

Solution. Proceeding as in above example calculate the E and L for each node as shown in the *Figure* 25.24.

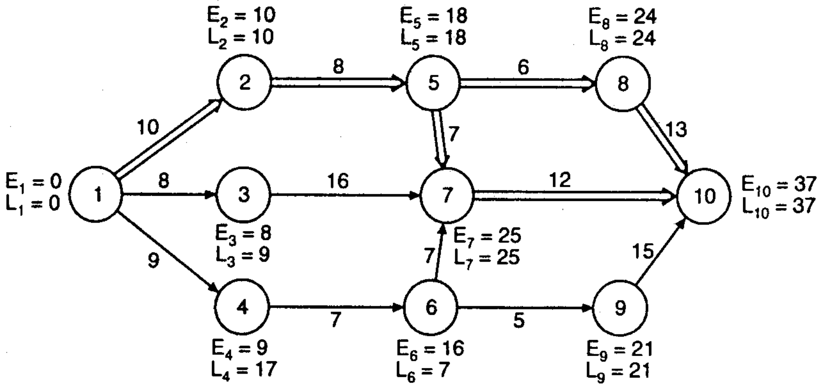


Fig. 25.24.

The early start T_E and late start T_L in respect of nod points is obtained in the following table.

[**Note.** Activity time in number of days are indicated on the network.]

Table 25.4. Network Analysis Table

Activity (i, j)	Normal Time (D_{ij})	Earliest Time		Latest Time		Float Time $(L_j - D_{ij}) - E_i$
		Start (E_i)	Finish $(E_i + D_{ij})$	Start $(L_j - D_{ij})$	Finish (L_j)	
(1, 2)	10	0	10	0	10	0
(1, 3)	8	0	8	1	9	1
(1, 4)	9	0	9	1	10	1
(2, 5)	8	10	18	10	18	0
(4, 6)	7	9	16	10	17	1
(3, 7)	16	8	24	9	25	1
(5, 7)	7	18	25	18	25	0
(6, 7)	7	16	23	18	25	2
(5, 8)	6	18	24	18	24	0
(6, 9)	5	16	21	17	22	1
(7, 10)	12	25	37	25	37	0
(8, 10)	13	24	37	24	37	0
(9, 10)	15	21	36	22	37	1

It is evident from the table that there are two possible critical paths :

(*i*) $1 \rightarrow 2 \rightarrow 5 \rightarrow 8 \rightarrow 10$, (*ii*) $1 \rightarrow 2 \rightarrow 5 \rightarrow 7 \rightarrow 10$.

Example 4. *Find the critical path and calculate the slack time for each event for the following PERT diagram.*

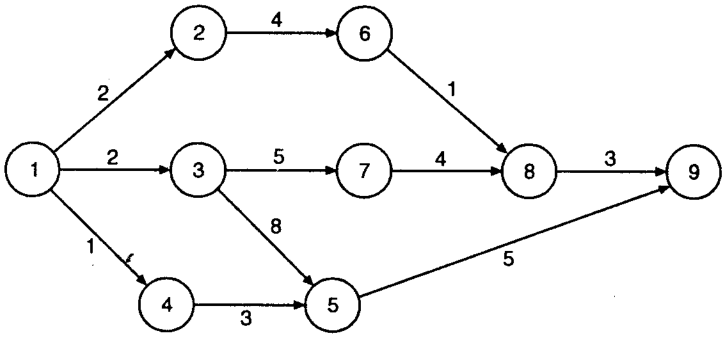


Fig. 25.25

Solution. Proceeding as in *Example 3*, the earliest times and latest times in respect of node points are obtained below :

Table 25.5 Network Analysis Table

Activity (i, j)	Normal Time (D_{ij})	Earliest Time		Latest Time		Float Available $(L_j - D_{ij}) - E_i$
		Start (E_i)	Finish $(E_i + D_{ij})$	Start $(L_j - D_{ij})$	Finish (L_j)	
(1, 2)	2	0	2	5	7	5
(1, 3)	2	0	2	0	2	0
(1,4)	1	0	1	6	7	6
(2, 6)	4	2	6	7	11	5
(3, 7)	5	2	7	3	8	1
(3, 5)	8	2	10	2	10	0
(4, 5)	3	1	4	7	10	6
(5, 9)	5	10	15	10	15	0
(6, 8)	1	6	7	11	12	5
(7, 8)	4	7	11	8	12	1
(8, 9)	3	11	14	12	15	1

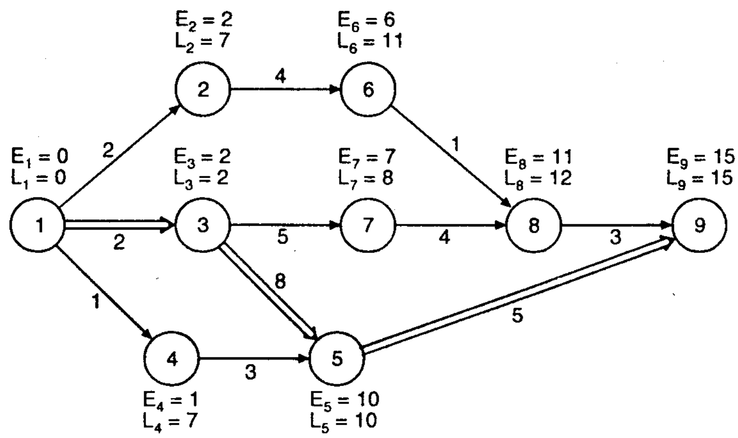


Fig. 25.26.

From above table critical nodes are the activities (1, 3), (3, 5) and (5, 9).
The critical path will be 1 → 3 → 5 → 9.

Example 5. *A project has the following times schedule :*

Activity	Time in weeks	Activity	Time in weeks
(1–2)	4	5–7	8
(1–3)	1	6–8	1
(2–4)	1	7–8	2
(3–4)	1	8–9	1
(3–5)	6	8–10	8
(4–9)	5	9–10	7
(5–6)	4		

Construct PERT network and compute :—
(i) T_E *and* T_L *for each event. (ii) Float for each activity.*
(iii) Critical path and its duration. **[Meerut (M. Com.) Jan. 98 BP]**

Solution. The network is constructed as given in *Fig.*25.27 :
The T_E's and T_L's computed on the network are as follows :

Event No.	:	1	2	3	4	5	6	7	8	9	10
T_E	:	0	4	1	5	7	11	15	17	18	25
T_L	:	0	12	1	13	7	16	15	17	18	25

Activity floats are computed below by using the formula :

$$Float = T_L\,(Head\ event) - T_E\,(Tail\ event) - Duration.$$

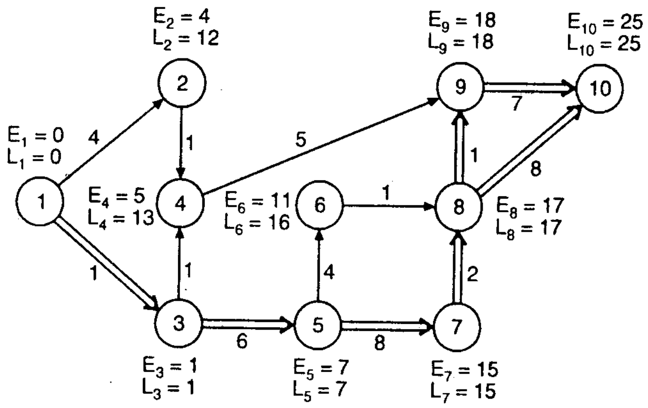


Fig. 25.27

Activity	Duration	T_E (Tail event)	T_L (Head event)	Float
(1–2)	4	0	12	8
(1–3)	1	0	1	0
(2–4)	1	4	13	8
(3–4)	1	1	13	11
(3–5)	6	1	7	0
(4–9)	5	5	18	8
(5–6)	4	7	16	5
(5–7)	8	7	15	0
(6–8)	1	11	17	5
(7–8)	2	15	17	0
(8–9)	1	17	18	0
(8–10)	8	17	25	0
(9–10)	7	18	25	0

Along the zero-float activities there are two critical paths as indicated by double line arrows in the network.

(*i*) 1 → 3 → 5 → 7 → 8 → 9 → 10 (*ii*) 1 → 3 → 5 → 7 → 8 → 10,

Example 6. *A project has the following time schedule* :

Activity	Time in months	Activity	Time in months
(1–2)	2	4–6	3
(1–3)	2	5–8	1
(1–4)	1	6–9	5
(2–5)	4	7–8	4
(3–6)	8	8–9	3
(3–7)	5		

Construct PERT network and comput : (i) Total float for each activity (ii) Critical path and its duration.

Also find the minimum number of cranes the project must have for its activities 2–5, 3–7 and 8–9 without delaying the project. Then, is there any change required in PERT network ? If so, indicate the name.

Solution. The network is constructed as below :

The network analysis table is computed below :

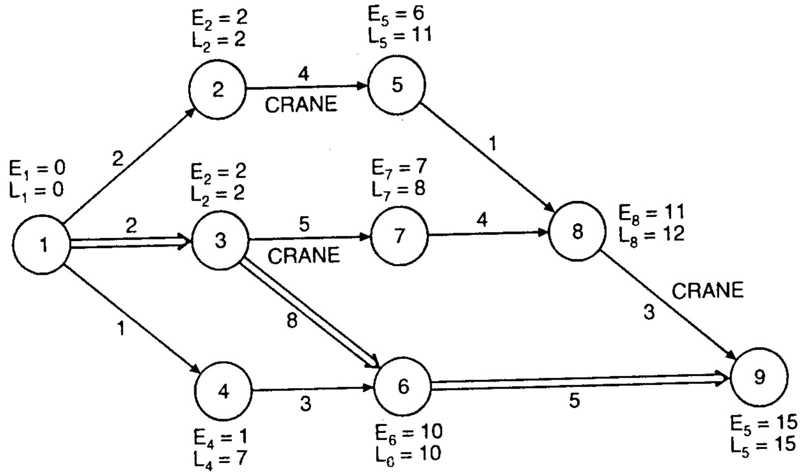


Fig. 25.28

Activity	Duration (D_{ij})	Earliest Time		Latest Time		Total Float
		Start (E_i)	Finish ($E_i + D_{ij}$)	Start ($L_j - D_{ij}$)	Finish (L_j)	($L_j - D_{ij}) - E_i$
(1–2)	2	0	2	4	7	5
(1–3)	2	0	2	0	2	0
(1–4)	1	0	1	6	7	6
(2–5)	4	2	6	7	11	5
(3–6)	8	2	10	2	10	0
(3–7)	5	2	7	3	8	1
(4–6)	3	1	4	7	10	6
(5–8)	1	6	7	11	12	5
(6–9)	5	10	15	10	15	0
(7–8)	4	7	11	8	12	1
(8–9)	3	11	14	12	15	1

Thus the critical path is 1 → 3 → 6 → 9 with duration 15 months.

Minimum number of cranes :

Finish (3–7) at 7	with one crane
Finish (2–5) at 7 + 4 = 11	with the same crane
Finish (5–8) at 11 + 1 = 12	without the crane
Finish (8–9) at 12 + 3 = 15	with the same crane

Therefore, one crane will be sufficient and activities (2—5), (5—8), and (8—9) will start at 7, 11, 12 respectively.

Example 7 (a). *A project schedule has the following characteristics :*

Activity	Time	Activity	Time
(1-2)	2	4-8	8
(1-4)	2	5-6	4
(1-7)	1	6-9	3
(2-3)	4	7-8	3
(3-6)	1	8-9	5
(4-5)	5		

(*i*) *Construct the PERT network and find critical path and time duration of the project.*

(*ii*) *Activities 2—3, 4—5, 6—9 each requires one unit of the same key equipment to complete it. Do you think availability of one unit of the equipment in the organization is sufficient for completing the project without delaying it ? If so, what is the schedule of these activities ?*

Solution. (*i*) Proceeding as in above examples compute the following network analysis table.

Activity $(i\text{–}j)$	Normal Duration (D_{ij})	Earliest Time		Latest Time		Total Float $(L_j - D_{ij}) - E_i$
		Start (E_i)	Finish $(E_i + D_{ij})$	Start $(L_j - D_{ij})$	Finish (L_j)	
(1-2)	2	0	2	5	7	5
(1-4)	2	0	2	0	2	0
(1-7)	1	0	1	6	7	6
(2-3)	4	2	6	7	11	5
(3-6)	1	6	7	11	12	5
(4-5)	5	2	7	3	8	1
(4-8)	8	2	10	2	10	0
(5-6)	4	7	11	8	12	1
(6-9)	3	11	14	12	15	1
(7-8)	3	1	4	7	10	6
(8,9)	5	10	15	10	15	0

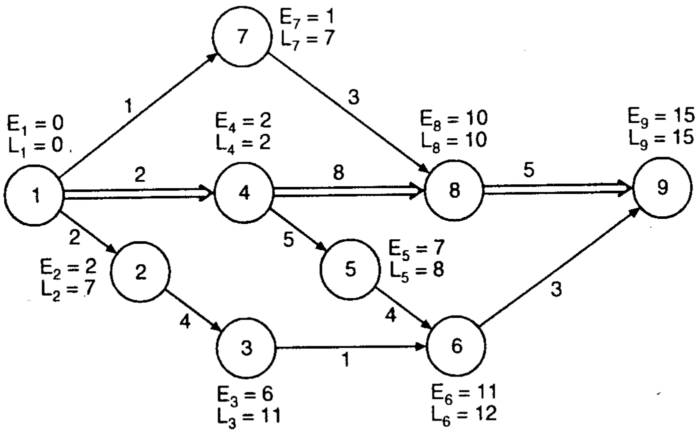


Fig. 25.29 (a)

Hence critical path (along 0 floats) is 1 → 4 → 8 → 9 with duration of 15.

(*ii*) Activity (6—9) can only be undertaken when both (2—3) and (4—5) and their following activities are over. Thus (2—3) and (4—5) contend for the equipment. The two alternative schedules for these are :

Activity	Start	Activity	Start
(2-3)	2	(4-5)	2
(4-5)	6	(2-3)	7
(6-9)	15	(6-9)	12

The second alternative does not delay the project completion time and hence to be recommended.

Example 7 (b). *A project has the following time schedule :*

Activity	Time in weeks	Activity	Time in weeks
(1–2)	4	(5–7)	8
(1–3)	1	(6–8)	1
(2–4)	1	(7–8)	2
(3–4)	1	(8–9)	1
(3–5)	6	(8–10)	8
(4–9)	5	(9–10)	7
(5–6)	4		

Construct a PERT network and compute :

(i) T_E and T_L *for each event* **[C.A. (May) 2000]**

(ii) Float for each activity, and

(iii) Critical path and its duration.

Solutioin. The network is conducted as given below :

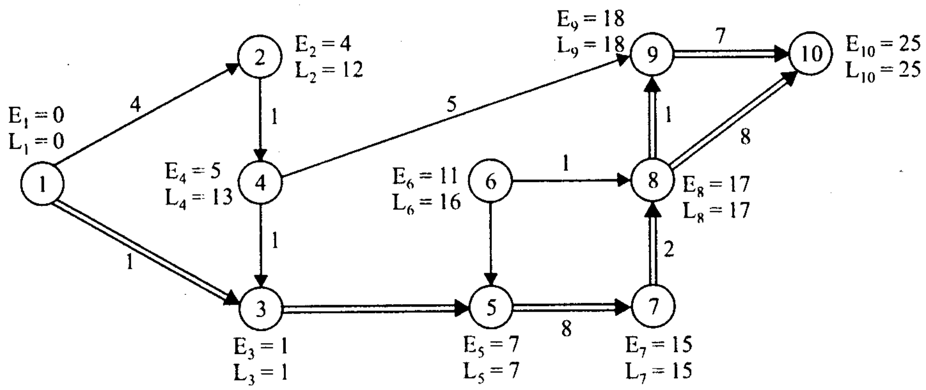


Fig. 25.29 (b)

(i) The T_E's and T_L's for various events constructed on the network are as follows :

Event No. :	1	2	3	4	5	6	7	8	9	10
T_E :	0	4	1	5	7	11	15	17	18	25
T_L :	0	12	1	13	7	16	15	17	18	25

(ii) Activity floats are computed using the following formula :

Float = T_L (Head event) – T_E (Tail event) – Duration.

Activity	Duration	T_E (Tail event)	T_L (Head event)	Float
(1–2)	4	0	12	8
(1–3)	1	0	1	0
(2–4)	1	4	13	8
(3–4)	1	1	13	11
(3–5)	6	1	7	0
(4–9)	5	5	18	8
(5–6)	4	7	16	5
(5–7)	8	7	15	0
(6–8)	1	11	17	5
(7–8)	2	15	17	0
(8–9)	1	17	18	0
(8–10)	8	17	25	0
(9–10)	7	18	25	0

Critical path is given by all those activities which have zero floats. Along the zero float activities, there are two such critical paths :

(i) $1 \to 3 \to 5 \to 7 \to 8 \to 9 \to 10$ (ii) $1 \to 3 \to 5 \to 7 \to 8 \to 10$

The project duration is 25 weeks.

Example 7 (c). *Consider a project consisting of the following jobs.*

Job	Predecessor	Time in days
A	—	15
B	—	10
C	A, B	10
D	A, B	10
E	B	5
F	D, E	5
G	C, F	20
H	D, E	10
I	G, H	15

Draw the network and determine the project duration. Also identify the critical path.

[AIMS (Bang.) MBA 2002]

25.9. EXAMPLES ON OPTIMUM DURATION AND MINIMUM DURATION COST

Example 8. *Table below shows, jobs, their normal time and cost, and crash time and cost for a project.*

Job	Normal Time (days)	Cost (Rs.)	Crash Time (days)	Crash Cost (Rs.)
(1–2)	6	1400	4	1900
(1–3)	8	2000	5	2800
(2–3)	4	1100	2	1500
(2–4)	3	800	2	1400
(3–4)	*Dummy*	-	-	-
(3–5)	6	900	3	1600
(4–6)	10	2500	6	3500
(5–6)	3	500	2	800

Indirect cost for the project is Rs. 300 per day.

(*i*) *Draw the network of the project*

(*ii*) *What is the normal duration cost of the project ?*

(*iii*) *If all activities are crashed, what will be the project duration and corresponding cost ?*

(*iv*) *Find the optimum duration and minimum project cost.*

Solution. (*i*) Network is shown in the *Fig.* 25.30.

(*ii*) Assuming that all activities occur at normal times, the critical path calculations are shown in the figure under normal conditions. The critical path is $1 \to 2 \to 3 \to 4 \to 6$. The duration of the project is 20 days and its associated (normal) cost is Rs. 9200.

(*iii*) Now compute the different minimum cost schedule that can occur between normal and crash times mainly depending on the cost time slopes for the different activities. To calculate these use the formula

$$\textbf{Cost Slope} = \frac{\textbf{Crash cost–Normal cost}}{\textbf{Normal time–Crash time}}.$$

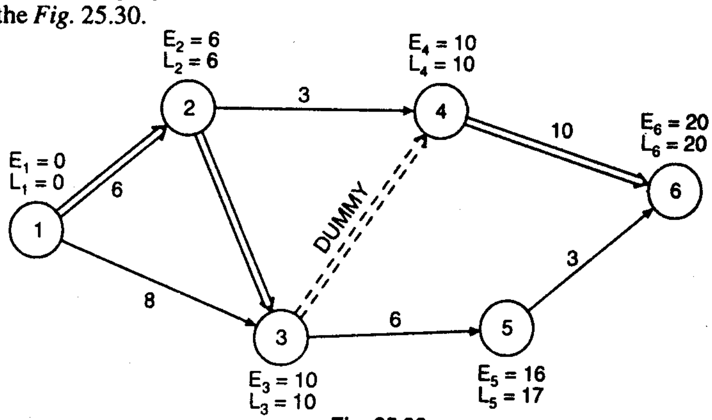


Fig. 25.30

These slopes are summarized in the following tabuar form :

Activity	:	(1–2)	(1–3)	(2–3)	(2–4)	(3–5)	(4–6)	(5–6)
Slope	:	250	267	200	600	233	250	300

Step 1. Because the present schedule involves more time, the schedule is reduced by crashing some of the activities. As the activities lying on the critical path control the duration of the project, therefore the duration of some activities lying on the critical path is reduced.

Start reducing the duration of that activity which involves minimum cost slope. As the activity (2–3) has the minimum cost slope, the duration of this acitivity is reduced from 4 to 2 days resulting an additional cost of Rs. 2 × 200 = Rs.400. But this activity should be shortened only by one day, since the path 1 → 2 → 4 → 6 becomes a parallel critical path. So the revised schedule corresponds to 19 days with a cost of Rs. (9200 + 200) = Rs.9400.

Step 2. Now it is evident that the activities (1–2) and (4–6) among the remaining activities lying on the two critical paths have the least slope. Therefore, either (1–2) or (4–6) can be compressed only for two days. This is due to the fact that 1 → 3 → 5 → 6, 1 → 2 →3 → 4 → 6, and 1 → 2 → 4 → 6, becomes three parallel critical paths. So three alterative choices are given below :

(*i*) Compress (1–2) by 2-days at a cost of Rs. 250.

(*ii*) Compress (4–6) by 2-days at a cost of Rs. 250.

(*iii*) Compress (1–2) and (4–6) by 1-day at a cost of Rs. 250 each.

The additional cost thus will be Rs. 2 × 250 = Rs.500. Thus a 17 days least cost schedule is obtained with a cost of Rs. (9400 + 500) = Rs.9900.

(*iv*) To determine the optimum schedule, compute the total cost by adding, the indirect cost corresponding to each schedule to the cost of crashing (slope) optimum schedule (duration) is then obtained for which the total cost is least. Required calculations are put in the following tabular form :

Normal Project length (days)	Crashing time and cost (days/Rs.)	Indirect cost @ Rs. 300	Total cost (Rs.)
20	–	20 × 300	6000
19	1 × 200 = 200	19 × 300	5900
18	1 × 250 = 250	18 × 300	5650
17	1 × 250 = 250	17 × 300	5350
16	1 × 200 + 1 × 600 + 1 × 233 = 1033	16 × 300	5833

Since minimum total cost is Rs. 5350, the optimum duration of the project is 17 days.

Example 9. *The following table gives the acitivities in a construction project and other relevent information.*

Acitivity (*i–j*)	Preceding activity	Normal time (days)	Crash time (days)	Normal cost (Rs.)	Crash cost (Rs.)
(1–2)	–	20	17	600	720
(1–3)	–	25	25	200	200
(2–3)	(1–2)	10	8	300	440
(2–4)	(1–2)	12	6	400	700
(3–4)	(1–3), (2–3)	5	2	300	420
(4–5)	(2–4), (3–4)	10	5	300	600

(*a*) *Draw the acitivity network of the project,*

(*b*) *Find the total float and free float for each activity,*

(*c*) *Using the above information "crash" or shorten the activity step-by-step until the shortest duration is reached.* **[VTU (BE Mech.) 2002]**

Solution. (*a*) The network is shown in the following figure.

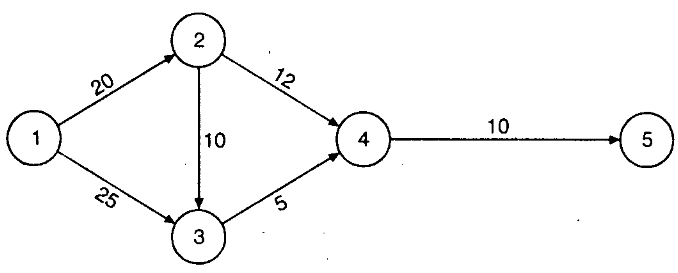


Fig. 25.31

(*b*) Considering the normal time of the project, the earliest and latest times as well as the total float, and free floats in respect of node points is put in the following table.

Acitivity $(i-j)$	Normal time (days) (D_{ij})	Start		Finish		Total float	Free float
		Earliest (E_i)	Latest $(L_j - D_{ij})$	Earliest $(E_i + D_{ij})$	Latest (L_j)		
(1–2)	20	0	0	20	20	0	0
(1–3)	25	0	5	25	30	5	5
(2–3)	10	20	20	30	30	0	0
(2–4)	12	20	23	32	35	3	3
(3–4)	5	30	30	35	35	0	0
(4–5)	10	35	35	45	45	0	0

The above table shows that the critical path is $1 \to 2 \to 3 \to 4 \to 5$ which is shown in the following figure.

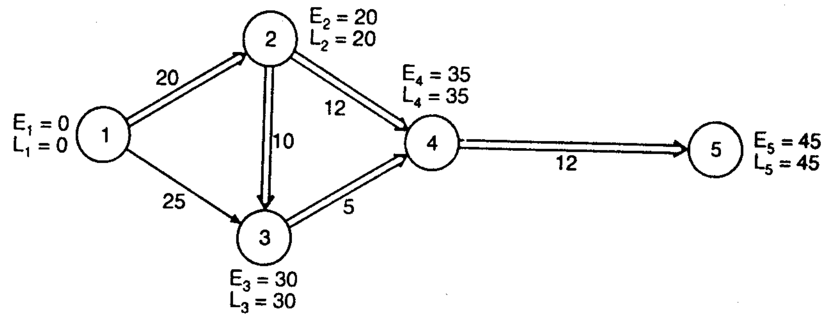


Fig. 25.32.

(*c*) Compute the different minimum cost schedule that can occur between normal and crash times, which are dependent on the cost time slopes for different activities. These are computed by the formula :

$$\textbf{Cost Slope} = \frac{\textbf{Crash cost} - \textbf{Normal cost}}{\textbf{Normal time} - \textbf{Crash time}}.$$

The slopes of the activities of above network are calculated as follows.

Activity	:	(1, 2)	(1, 3)	(2, 3)	(2, 4)	(3, 4)	(4, 5)
Slope	:	40	0	70	50	40	60

Step 1. Since the project duration is controlled by the activities on the critical path, the duration of some activities lying on the critical path is reduced.

First, the duration of that activity is reduced which has the minimum cost slope. Since the activities (1, 2) and (3, 4) both give the minimum cost slope, the duration of acitivity (3, 4) is compressed from 5 to 2 days with an additional cost of Rs. 3×40 = Rs.120. It should be noted that we have chosen the activity (3, 4) because in doing so two parallel critical paths $1 \to 2 \to 4 \to 5$ and $1 \to 2 \to 3 \to 4 \to 5$ are obtained. Thus the revised schedule corresponds to 42 days with a cost of Rs. (2100 + 120) = Rs.2220.

Step 2. Further if the duration of any activity of critical path is reduced, the two critical paths of the project remain the same.

Therefore, since all the activities on the critical path $1 \to 2 \to 3 \to 4 \to 5$ are at crash time, it is no longer possible to compress the time of the project. Hence the minimum duration of the project will be 42 days.

Example 10. *A small marketing project consists of the jobs in the table given below, With each job is listed its normal time and a minimum or crash time (in days). The cost in (Rs. per day) of crashing each job is also given.*

Job (i–j)	Normal duration (in days)	Minimum (crash) duration (in days)	Cost of crashing (Rs. per day)
(1–2)	9	6	20
(1–3)	8	5	25
(1–4)	15	10	30
(2–4)	5	3	10
(3–4)	10	6	15
(4–5)	2	1	40

(*a*) *What is the normal project length and the minimum project length.*

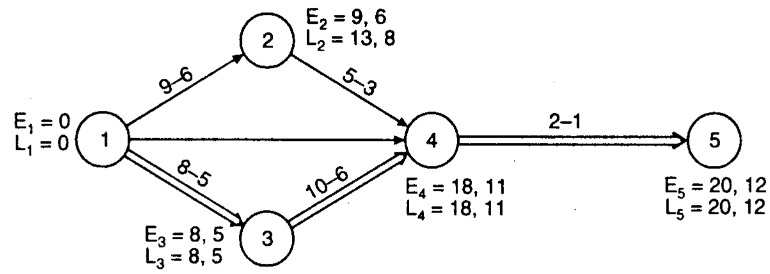


Fig. 25.33

(*b*) *Determine the minimum crashing costs of schedules ranging from normal length down to, and including, the minimum length schedule, i.e. if L is the length of the normal schedule, find the costs of schedules which are L, L – 1, L – 2, and so on, long days.*

(*c*) *Overhead costs total Rs. 60 per day. What is the optimal length schedule duration of each job for your solution.* **[Tamilnadu (ERODE) 97]**

Solution. (*a*) Construct the network considering the normal duration of the project as given below.

The critical path is $1 \to 3 \to 4 \to 5$ with the normal duration 20 days and minimum project length is 12 days.

(*b*) Since the present schedule consumes more time, the duration can be reduced by crashing some of the activities. Also, since the project duration is controlled by the activities lying on the critical path, the duration of some of the activities lying on the critical path can be reduced.

Step 1. First, reduce the duration of that activity which involves the minimum cost. Since the activity (3, 4) involves minimum cost, the duration of this activity can be compressed from 10 days to 9 days resulting on total cost for 19 day's schedule becomes = *Rs.* 15 + Rs. 19 × 60 = Rs. 1155.

Step 2. Again since the critical path remains unchanged, the duration of acitvity (3, 4) can be reduced further from 9 days to 8 days resulting in an additional cost of Rs. 2 × 15, *i.e.* Rs. 30. So the total cost for 18 days schedule becomes = Rs. 30 + Rs. 18 × 60 = Rs.1110

Step 3. Continue this procedure till the total cost starts increasing. The calculations may be compiled in the following table :

Normal project length (days)	Crashing cost (days/Rs.)	Overhead @ Rs. 60/day	Total cost (Rs.)
20	—	20 × 60	1200
19	1 × 15 = 15	19 × 60	1155
18	2 × 15 = 30	18 × 60	1110
17	3 × 15 = 45	17 × 60	1065
16	3 × 15 + 1 × 40 = 85	16 × 60	1045
15	4 × 15 + 1 × 40 + 1 × 30 = 130	15 × 60	1030
14	4 × 15 + 1 × 40 + 2 × 30 + 1 × 25 + 1 × 10 = 195	14 × 60	1035

(*c*) Since the total cost starts increasing for 14 days duration, the minimum total cost of Rs. 1030 occurs for 15 days duration. Hence the optimum length of the schedule is 15 days.

Optimum duration of each job is as follows :

Job	:	(1,2)	(1,3)	(1,4)	(2,4)	(3,4)	(4,5)
Optimum Duration (days)	:	9	8	14	5	6	1

Example 11. *In the project network shown in the figure given below, the nodes are denoted by numbers and the activities by letters. The normal and crash durations of the various activities along with the associated costs are shown below :*

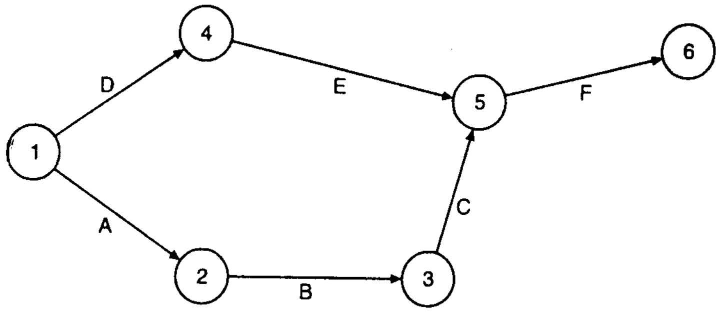


Fig. 25.34

Activity	Normal duration (days)	Normal cost (Rs.)	Crash duration (days)	Crash cost (Rs.)
A	8	1800	6	2200
B	16	1500	11	2200
C	14	1800	9	2400
D	12	2400	9	3000
E	15	800	14	2000
F	10	2000	8	4000

Determine the least cost 36 days schedule.

Solution. First assume that all activities occur at normal times. Then the following network shows the critical path computations under normal conditions. The critical path is $A \rightarrow B \rightarrow C \rightarrow F$. The schedule of the project is 48 days and its associated normal cost becomes

= Rs.(1800 + 1500 + 1800 + 2400 + 800 + 2000) = Rs.10,300.

The different minimum cost schedule that can occur between normal and crash times, which are mainly dependent on the *cost time slopes* for different activities. The cost time slopes can be computed by the formula :

$$\textbf{Cost – time Slope} = \frac{\textbf{Crash cost–Normal cost}}{\textbf{Normal time–Crash time}}$$

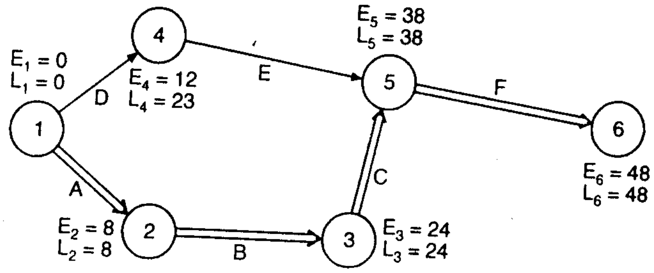


Fig. 25.35

These slopes for the activities of the above network are obtained as follows :

Activity	:	A	B	C	D	E	F
Slope	:	200	140	120	200	1200	1000

Now proceed step-by-step as follows :

Step 1. Since the present schedule consumes more time, the schedule can be reduced by crashing some of the activities. Since the project duration is controlled by the activities lying on the critical path, the duration of some activities on the critical path is reduced.
First reduce the duration of that activity which involves minimum cost. Activity C with minimum slope gives the minimum cost. So the duration of activity C is compressed from 14 days to 9 days with an additional cost Rs.5 × 120 = Rs.600. Therefore, new schedule corresponds to 43 days with a cost of Rs.(10300 + 600) = Rs.10900.

Step 2. Now, it can be observed that the present schedule still consumes more time and also not all the activities on the critical path are at their crash durations. Hence the project duration can be reduced by crashing some other activity. Out of the remaining activities on the critical path, the activity B has the least slope. So reduce the duration of activity B from 16 days to 11 days at a cost of Rs. = 5 × 140 = Rs.700. Thus the new project duration becomes 38 days with a cost of Rs.(10900 + 700) = Rs.11600.

Step 3. This project duration is still more than the required duration of 36 days. So select some other activity lying on the critical path for crashing. Obviously, only the activities A and F on the critical path can be considered for crashing. Since activity A has the smaller slope, the duration of A can be compressed. Compress A by only one day although it can be compressed by 2 days (from 8 to 6 days). Because, the path $1 \to 4 \to 5 \to 6$ becomes a parallel critical path as soon as A is compressed by one day. Thus a new schedule corresponds to 37 days with a cost of Rs,(11600 + 200) = Rs.11800.

Step 4. Since only 36 days schedule is required, compress some activity by one day. To do so compress one day in each of the two parallel critical paths. So there are three choices :
(*i*) Activity F can be compressed by one day at a cost of Rs. 1000.
(*ii*) Activities A and D can be compressed by one day each (since B and C are already at their crash points). This gives the total cost of Rs. (200 + 200) = Rs.400.
(*iii*) Activities A and E can be compressed by one day each at a total cost of Rs. (200 + 1200) = Rs..1400.

But, the second choice gives the least cost schedule and hence it should be selected. This involves a 36 days schedule with a cost of Rs.(11800 + 400) = Rs.(12,200).

Example 12. *Determine the least cost schedule from the following project using CPM technique. Overhead cost per day is Rs. 6.*

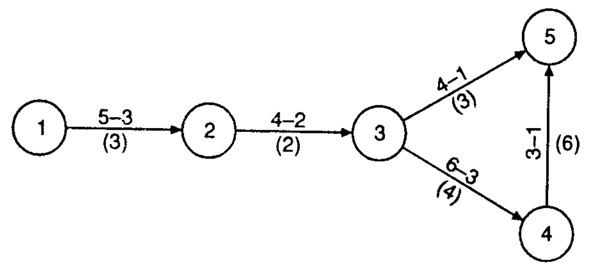


Fig. 25.36.

The numbers above and below the activities have their usual meaning.

Solution. Taking into account the normal duration of the project, the critical path is found as $1 \to 2 \to 3 \to 4 \to 5$.

In order to determine the least cost schedule, compress the duration by crashing some of the activities. Since the activities lying on the critical path control the project duration, the duration of some activities lying on the critical path can be shortened.

The following table gives the normal length, crash cost, and total cost.

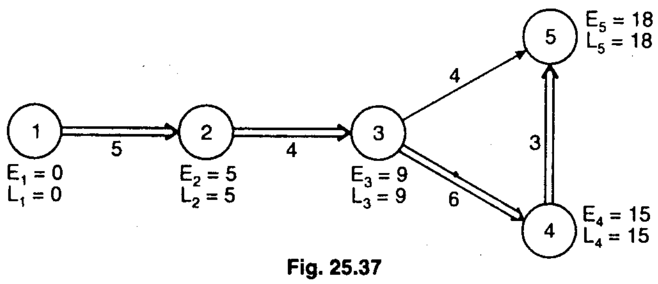


Fig. 25.37

Normal project length (days)	Crashing cost (Rs.)	Overhead cost @ Rs. 6 per day	Total cost (Rs.)
18	–	18 × 6	108
17	1 × 2 = 2	17 × 6	104
16	2 × 2 = 4	16 × 6	100
15	2 × 2 + 1 × 3 = 7	15 × 6	97
14	2 × 2 + 2 × 3 = 10	14 × 6	94
13	2 × 2 + 2 × 3 + 1× 4 = 14	13 × 6	92
12	2 × 2 + 2 × 3 + 2 × 4 = 18	12 × 6	90
11	2 × 2 + 2 × 3 + 3 × 4 = 22	11 × 6	88
10	2 × 2 + 2 × 3 + 3 × 4 + 1 × 6 = 28	10 × 6	88
9	2 × 2 + 2 × 3 +3 × 4 + 2 × 6 = 34	9 × 6	88

Since there are two parallel critical paths when the project length is 9 days and also the activities are at their crash time, the optimum length of the schedule is 9 days with total cost of Rs. 88.

Example 13. *The time and cost estimates and precedence relationship of the different activities constituting a project are given below :*

Activity	Predecessor activity	Time (in weeks)		Cost (in rupees)	
		Normal	Crash	Normal	Crash
A	None	3	2	8,000	9,000
B	None	8	6	600	1,000
C	B	6	4	10,000	12,000
D	B	5	2	4,000	10,000
E	A	13	10	3,000	9,000
F	A	4	4	5,000	5,000
G	F	2	1	1,200	1,400
H	C, E, G	6	4	3,500	4,500
I	F	2	1	700	800

(i) Draw a project network diagram and find the critical path.

(ii) If a dead line of 17 weeks is imposed for the completion of the project, what activities will be crashed ? What would be the additional cost and the critical activities after crashing the project ? **[Delhi (M. Com.) 97]**

Solution. (i) The network with the normal and the shortest (crash) times for various activities is shown below :

The critical path of the project is 1–2–5–6 (or *A–E–H*) with a length of 22 weeks.

(ii) To complete the project in the stipulated 17 weeks, crashing would have to be done by determining the weekly crashing cost for each of the activities using the formula :

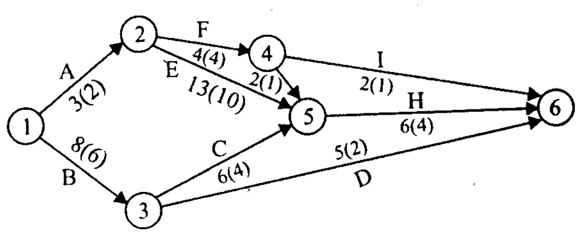


Fig. 25.38.

$$\text{Crashing cost per week} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}$$

Activity :	A	B	C	D	E	F	G	H	I
Node :	(1—2)	(1—3)	(3—5)	(3—6)	(2—5)	(2—4)	(4—5)	*5—6)	(4—6)
Crashing cost (Rs.) :	1,000	200	1,000	2,000	2,000	—	200	500	1,000

Activities to be crashed and the cost involved are shown in table below :

Table : Crashing Schedule of the Project

Crashing	Alternative	Cost	Decision	Duration	Critical Path(s)
First	(i) *A* (1—2) (ii) *E* (2—5) (iii) *H* (5—6)	1,000 2,000 500	Crash (5—6)	21	1—2—5—6
Second	(i) *A* (1—2) (ii) *E* (2—5) (iii) *H* (5—6)	1,000 2,000 500	Crash (5—6)	20	1—2—5—6
Third	(i) *A* (1—2) (ii) *E* (2—5)	1,000 2,000	Crash (1—2)	19	1—2—5—6
Fourth	(i) *E* (2—5)	2,000	Crash (2—5)	18	1—2—5—6
Fifth	(i) *E* (2—5) *B* (1—3) (ii) *E* (2—5) *C* (3—5)	2,200 3,000	Crash (2—5), (1—3)	17	1—3—5—6 1—2—5—6 1—3—5—6

(a) Activities to be crashed : *A*—one week, *B*—one week, *E*—two weeks and *I*—two weeks.

(b) Total cost of the project = Normal cost + Crashing cost

= (18,000 + 600 + 10,000 + 4,000 + 3,000 + 15,000 + 1,200 + 3,500 + 7,000)

+ (500 + 500 + 1,000 + 2,000 + 2,200) = Rs. 68,500

(c) Activities which are critical after crashing are *A*, *B*, *C*, *E* and *H*.

EXAMINATION PROBLEMS

1. (a) A directed graph has the lines (*a*, *b*), (*a*, *c*), (*c*, *a*), (*a*, *d*) and (*d*, *c*). Draw the graph and say from which node there is a path to each other node and from which node there is no path.

[**Ans.** There is no path from node *b*. The possible paths from other nodes are :

a → *d* → *c* → *a*, *d* → *c* → *a* → *d*, *d* → *c* → *a* → *b*]

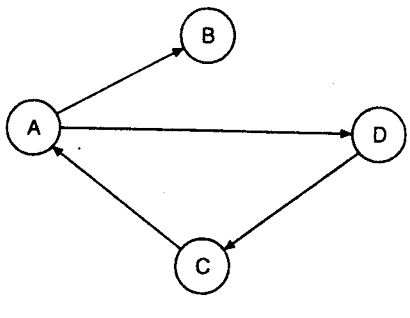


Fig. 25.39.

(b) A small projet has the following time and cost estimates :

Job	Immediate Predecessor	Normal		Minimum	
		Time (Hrs.)	Cost (Rs.)	Time (Hrs.)	Cost (Rs.)
A	—	8	80	6	100
B	A	7	40	4	94
C	A	12	100	5	184
D	A	9	70	5	102
E	B, C, D	6	50	6	50

The overhead costs are Rs. 25 per hour. Compute the minimum cost project schedule. **[VTU (BE Mech.) 2003]**

2. Draw a network for the following project and number the events according to *Fulkerson's rule* :

(*i*) A is the start event and *K* is the end event.
(*ii*) *J* is the successor event to *F*.
(*iii*) *C* and *D* are successor events to *B*.
(*iv*) *D* is the preceding event in *G*.
(*v*) *E* and *F* occur after *C*.
(*vi*) *E* precedes *F*.
(*vii*) *C* restrains the occurrence of *G* and *G* precedes *H*.
(*viii*) *H* precedes *J*.
(*ix*) *F* restrains the occurrence of *H*.
(*x*) *K* succeds *J*.

[**Hint.** Number the nodes of the network such that their ascending order indicates the direction of progress in the network.]

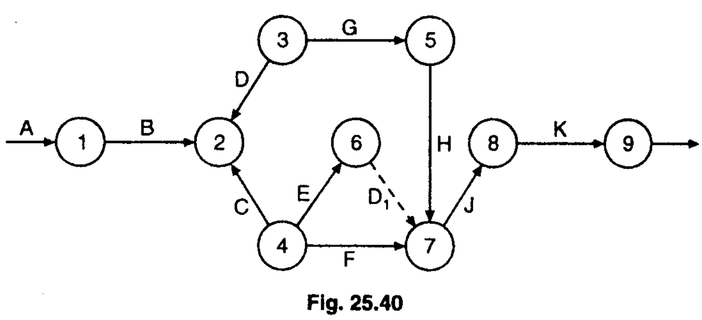


Fig. 25.40

3. Draw network (PERT) diagram from the following list of Activities.

Job Name	Immediate Predecessors	Job Name	Immediate Predecessors
a	–	*l*	k
b	*a*	m	k
c	b	n	k
d	c	o	d
e	b	p	o
f	e	q	b
g	e	r	n
h	c	s	l, n
i	c, f	t	s
j	g, h, i	u	p, q
k	j	v	u

[**Ans.** the network is drawn as below.]

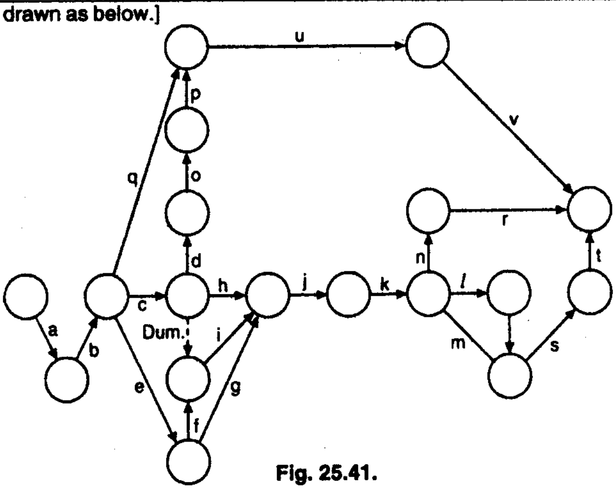


Fig. 25.41.

4. Draw network for jobbing production and indicate the critical path from the following :

Activity	Description	Time (Weeks)	Preceded by
A	Market research	15	–
B	Make drawings	15	–
C	Decide production policy	3	*A*
D	Prepare sales program	5	*A*
E	Prepare operation sheets	8	*B, C*
F	Buy materials	12	*B, C*
G	Plan labour force	1	*E*
H	Make tools	14	*E*
I	Schedule production	3	*D, G*
J	Produce product	14	*F, H, I*

[**Ans.** The network is drawn as shown below :

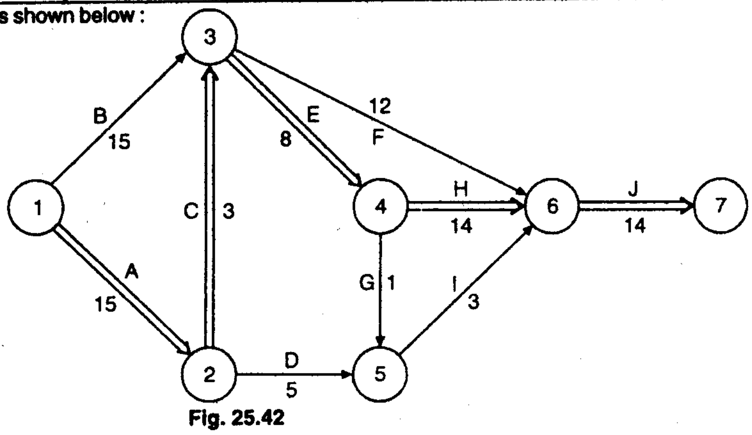


Fig. 25.42

5. The following information is known for a project. Draw the network and find the critical path. Capital letters denote activities and number in bracket denote activity times.

This must be completed	Before this can start	This must be completed	Before this can start
A(30)	*C*	F(7)	*H*
B(7)	*D*	F	I
B	G	F	L
B	K	G(21)	*I*
C(10)	*D*	F	L
C	G	H(7)	*J*(15)
D(14)	*E*	I (12)	J
E(10)	*F*	K(30)	*L*(15)

[**Ans.** The network is drawn as below. By enumerating the durations of the various paths it can be ascertained that the critical (longest) path is *A* → *C* → *D* → *E* → *F* → *I* → *J* with 98 days duration.]

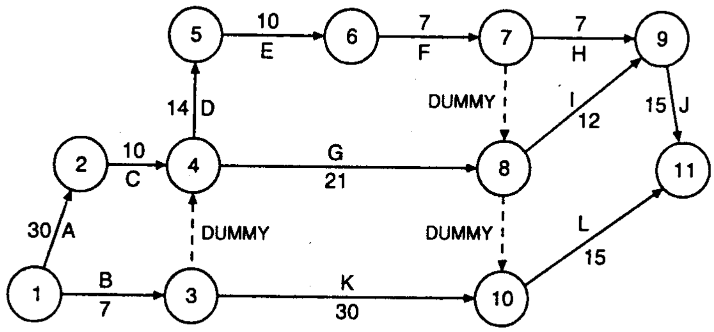


Fig. 25.43

6. For the following two projects, state the problem in terms of events and draw the event oriented networks.

(*a*) Conducting an examination

Design questionnaire	7 days
Print question paper	2 days
Distribute to various centres	4 days
Answer questionnaire	1 days
Collect answer books at main office	4 days

(*b*) Holding a conference

By mail ask members for suitable dates	6 days
Inform date to members	2 days
Prepare agenda	3 days
Send agenda and relevent materials to members by mail	7 days
Arrange conferece room	2 days
Arrange refreshments	1 days

[**Ans.** (a) *Examination*
1. Questionnaire prepared.
2. Question paper printed.
3. Papers distributed to various centres.
4. Questionnaire answered.
5. Answer-books collected at main office.

(b) *Conference*
1. Member's choice of date obtained.
2. Members informed of date.
3. Agenda prepared.
4. Agenda and relevent materials sent by mail to members.
5. Conference room arranged.
6. Refreshments arranged.

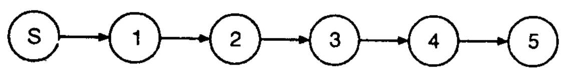


Fig. 25.44

Event 7 is the actual start of the conference. Activity 4—7 might represent the time for arrival of members from distant places.

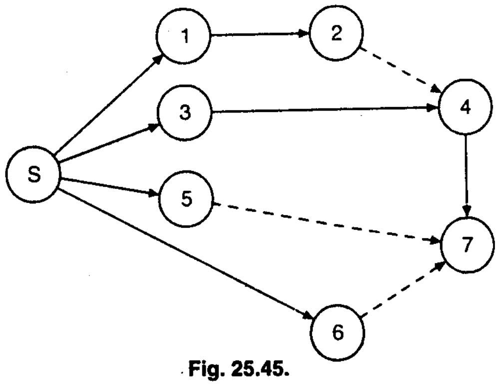


Fig. 25.45.

7. Draw the network diagram from the following activities and find critical path and total float of activities.

Job	Job time (days)	Immediate Predecessors
A	13	–
B	8	A
C	10	B
D	9	C
E	11	B
F	10	E
G	8	D, F
H	6	E
I	7	H
J	14	G, I
K	18	J

[**Ans.** Critical path 1 → 2 → 3 → 5 → 6 → 8 → 9 → 10 as shown in the following network diagram.

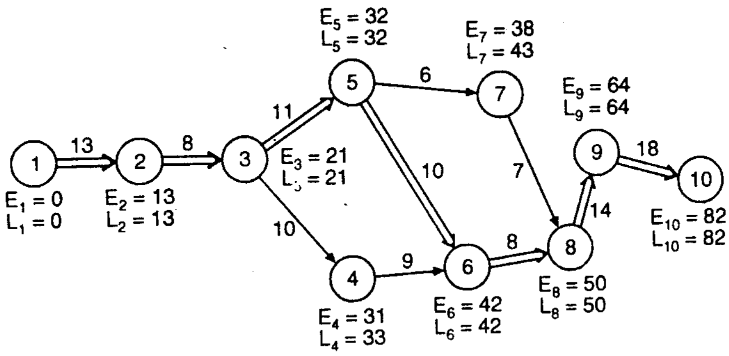


Fig. 25.46

Activity	:	A	R	C	D	E	F	G	H	I	J	K
Total Float ($T_L - T_E$)	:	0	0	0	0	5	0	0	0	2	0	0

8. Define an activity, event and dummy constraints in a PERT network. Using the following information plot a network. Determine the critical path and compute slack for all events :

Activity	Activity duration (in weeks)	Activity	Activity duration (in weeks)
(0–1)	5	(3–5)	7
(0–2)	10	(3–6)	11
(1–2)	4	(4–6)	8
(1–3)	8	(4–7)	9
(1–4)	3	(5–7)	9
(2–3)	6	(5–6)	4
(2–5)	8	(6–7)	1

If the duration of activity (5—6) is increased to 6 and activity (3—6) is reduced, what will be the new critical path ?

9. The project represented in the table below is to be scheduled within a resource limit of 12 men. All the men are capable of working on any of the jobs.If not assigned on a particular day, a man is idle but still draws pay. Each job must be assigned to a crew of men corresponding to one of the three possible crew sizes listed in the table. No. in between assignments may be made, the crew size must remain fixed for a job until it is finished. Job duration equals man-days divided by crew size for any crew size chosen. Schedule the project so as to minimize idle man-days over its active span.

Job *i–j*	Resource requirement *(Man-days)*	Crew size (men)		
		Minimum	Normal	Maximum
1–2	32	2	4	8
1–3	48	4	6	8
2–3	40	4	5	8
2–4	12	2	3	4
4–5	30	3	5	6
3–5	54	3	6	9

[**Ans.** Critical path is to do jobs 1 → 2 → 3 → 5.]

10. A contractor has received order for constructing a cottage on a sea side resort. The delivery of materials must be planned and the complete job finished in 13 weeks.

The work involves the following (the numbers represent number of days) :

Buying brickets and cement (8, 10, 14)
Roof tiles (20, 24, 30)
Repairing foundation (12, 14, 16)
Erecting shell structure of building (18, 20, 24)
Laying drains (12, 14, 15)
Plumbing (20, 24, 30)
Flooring (8, 10, 12)
Wiring for electricals (16, 20, 26)
Constructing roof (8, 8, 10)
Plastering (12, 12, 18)
Lamdscaping (4, 4, 6)
Painting and cleaning (10, 12, 14)
Laying pathway (4, 4, 4)
Installing doors and fittings (4, 4, 4)

Construct a logical PERT diagram and mark on it the critical path circling the nodes and indicating activity with arrow. What is total critical path time ? Can the completion target be met ?

11. The chairman of *ABC* consulting company has an opportunity to participate in a marketing project that has a sales price of Rs. 90,000 but must be completed within 8 weeks. The letter of indent was received Friday afternoon. Both the *Head of Marketing Department* and the *Cost Accountant* came on Saturday and completed the appropriate time and cost for you based upon past jobs. Since the chairman needs an answer at 8.30 A.M. on Monday (start of the 8-weeks), you have been requested to determine the profitability of the project on an 8 week basis. An answer at 8.30 A.M. Monday allows the firm to start the project at 10.00 A.M. in order to stay within the 8 weeks demanded by the customer. The time and cost under normal conditions without crashing the project is based upon an 11-week basis. What answer should the chairman give the customer on Monday morning ? A table of time and costs is given below :

Event	Preceding Event	Normal		Crash	
		Time (in weeks)	Cost (in Rs.)	Time (in weeks)	Cost (in Rs.)
4	1	2	8,000	1	13,000
2	1	3	7,000	1	13,000
3	1	6	11,000	5	13,500
4	2	4	6,000	3	10,000
3	2	2	9,000	1	10,000
5	2	7	8,500	6	11,500
5	4	4	10,500	3	16,000
5	3	3	5,000	2	7,000

12. (a) Discuss the benefits of Network Techniques in Project Planning and Control.

(b) The basic cost-time data for jobs in a project are as given below :

Job	Normal Time		Crash Time		Cost of Crashing per day
	Days	Cost (Rs.)	Days	Cost (Rs.)	
A	3	140	2	210	70
B	6	215	5	275	60
C	2	160	1	240	80
D	4	130	3	180	50
E	2	170	1	250	80
F	7	165	4	285	40
G	4	210	3	290	80
H	3	110	2	160	50
Total		1,500		1,890	

The activity (Job) dependencies are as below :
(*i*) *A, B, C* are starting activities.
(*ii*) Activities *D, E* and *F* can start when once *A* is completed.
(*iii*) Activity *G* can start after *B* and *D* are completed.
(*iv*) Activity *H* can start after *C* and *E* are completed.
(*v*) Activities *G, F* and *H* are the final activities.
(1) Draw the network and indicate the critical path.
(2) What is the total time required to complete the project ? (based on normal times).
(3) If the project is to be completed in 8 days, what is the minimum cost to be incurred ? Indicate this cheapest cost schedule.

13. Explain network problems. Describe the method of drawing network diagrams.

14. A project has the following details. The indirect cost of the project per week is Rs. 1000/-

Activity	Normal time (weeks)	Crash time (weeks)	Direct cost slop (Rs./week)
1–2	6	4	100
1–3	10	6	300
1–4	15	7	600
2–4	4	3	700
3–5	15	10	500
4–5	15	8	800

(i) Draw the network and find the normal duration of the project with normal total cost.
(ii) Determine the optimal duration and its corresponding cost.
(iii) If all the activities are crashed to their maximum values, determine the duration and total cost of project.

[AIMS (Bangl.) MBA 2002]

15. Construct the network diagram comprising activities B, C ... Q and V such that the following constraints are satisfied :

B < E, F; C > G, L; E, G < H;
L, H < I; L < M ; H < N ;
H < J ; I, J < P ; P < Q. **[Bhubnashwar (IT) 2004]**

25.10 PROJECT EVALUATION AND REVIEW TECHNIQUE (PERT)

In the network analysis discussed so far, it is implicitly assumed that the time values are deterministic or variations in time are insignificant. This assumption is valid in regular jobs such as maintenance of a machine, etc., construction of a building or road, planning for production, as these are done from time to time and various activities could be timed very well. However, in research projects or design of a gear box of a new machine, various activities are based on judgement. A reliable time estimate is difficult to get because the technology is changing repidly. Time values are subject to chance variations.

The main objective in the analysis through PERT is to find out the completion for a particular event within specified date. If yes, what are the chances of completing the job ? The PERT approach takes into acount the uncertainties. In this approach, three time values are associated with each activity : the *optimistic value*, the *pessimistic value*, and the *most likely value*. These three time values provide a measure of uncertainty associated with that activity.

Def. 1. The ***optimistic time*** *is the shortest possible time in which the activity can be finished. It assumes that every thing goes very well. This is denoted by* t_0. **[Bhubneshwar (IT) 2004]**

Def. 2. *The* ***most likely time*** *is the estimate of the normal time the activity would take. This assumes normal delays. If a graph is plotted in the time of completion and the frequencey of completion in that time period, then the most likely time will represent the highest frequency of occurrence. This is denoted by* t_m.

Def. 3. *The* ***pessimistic time*** *represents the longest time the activity could take if everything goes wrong. As in optimistic estimate, this value may be such that only one in hundred or one in twenty will take time longer than this value. This is denoted by* t_p.

[Bhubneshwar (IT) 2004]

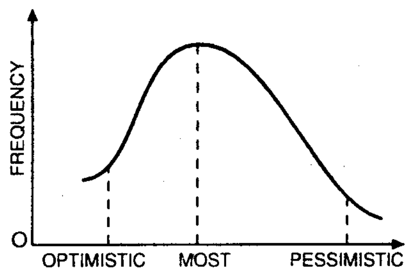


Fig. 25.47. Time distribution curve.

These three time values are shown in *Fig.* 25.46.

In order to obtain these values, one could use time values available for similar jobs, but most of the time the estimator may not be so fortunate to have this data. Secondly, values are the functions of manpower, machines and supporting facility. A better approach would be to seek opinion of 'experts in the field' keeping in view the resources available.

This estimate does not take into account such natural catastrophes as fire, etc.

In PERT calculation, all values are used to obtain the per cent expected value.

Def. 4. ***Expected time*** *is the average time an activity will take if it were to be repeated on large number of times and is based on the assumption that the activity time follows Beta distribution*. This is given by the formula :*

$$t_e = (t_o + 4t_m + t_p)/6.$$

Def. 5. *The* ***variance*** *for the activity is given by the formula :*

$$\sigma^2 = [(t_p - t_o)/6]^2,$$

where t_o is the *optimistic* time, t_p is the *pessimistic* time, t_m is the *most-likely* time, t_e is the *expected* time and σ^2 is the *variance*.

PERT computations are essentially the same as used earlier.

Q. 1. Explain the following terms in PERT : **[Bhubneshwar (IT) 2004]**
(*i*) Optimistic time, (*ii*) Normal time, (*iii*) Pessimistic time, (*iv*) Expected time, (*v*) Variance in relation to activities.

2. What are the requirements for the application of PERT ? Give an algorithm for PERT and state the limitations of this technique. **[Meerut (OR) 2003]**

The main difference is that instead of activity duration, expected time t_e for the activity is considered. With each node, variance is associated. Thus, the duration of the project is the mean expected time with variance.

Consider the network of *Fig* 25.20 again. *Table* 25.6 give three time estimates for each activity, the expected value and the variance also.

Table 25.6

Activity	t_o	t_m	t_p	$t = (t_o + 4t_m + t_p)/6$	$\sigma^2 = [(t_p - t_o)/6]^2$
(1–2)	1.0	2.00	3.0	2	4/36
(1–3)	1.5	2.00	2.5	2	1/36
(1–4)	1.5	2.75	3.5	3	4/36
(2–5)	3.0	3.00	7.0	4	16/36
(3–6)	4.0	4.50	8.0	5	16/36
(3–7)	6.0	8.25	9.0	8	9/36
(4–7)	3.0	3.50	7.0	4	16/36
(5–8)	2.0	2.00	2.0	2	0
(6–8)	2.0	4.00	6.0	4	16/36
(7–9)	2.0	4.50	8.0	5	36/36
(8–9)	2.0	3.00	4.0	3	4/36
(9–10)	2.5	4.25	4.5	4	4/36

Once, expected values have been calculated, these are used in finding the critical path. In this particular example, three estimates are so chosen that mean values are same as before and hence critical path calculations are same as before. However, the interpretation of the critical path is now different. In this case, the expected duration of job taken time less than 9 days or more than 19 days too. Then, meaning of the expected duration is that—if the same job is performed again under similar conditions, the average duration will be 19 days. If the job takes 19 days, then there is probability value associated with it which can be calculated under some assumptions. Since duration of each activity is a random variable, the duration of a path which consists of a set of activities will also be a random variable. To calculate the exact distribution of the duration of a path will be difficult and for management decisions

---

* The Beta distribution was chosen possibily because it is a
(1) Unimodal distribution
(2) Finite non-negative end points
(3) Non-symmetric or symmetric Beta density is given by

$$f(x, \alpha, \beta) = \frac{1}{B(\alpha+1, \beta+1)} x^{\alpha} (1-x)^{\beta}, \quad \begin{cases} 0 < x < 1 \\ \alpha > -1, \beta > -1 \end{cases}.$$

If x takes on values between limits *a* and *b*, a new variable $y = (x - a)/(b - a)$ can be defined and this takes values between zero and one. Beta function is defined by

$$B(\alpha, \beta) = \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)}, \text{ where } \Gamma\alpha \text{ is "gamma } \alpha\text{".}$$

it is enough to know the mean and the variance. Mean value has been calculated using the method discussed earlier. The same approach is used to find the variance.

Rules for finding variance of events.

(*i*) Variance for the initial event is zero. Set $V_1 = 0$.

(*ii*) V_j, the variance for succeeding event j in question is obtained by adding activities variance to the variance of predecessor event except at merge points, *i.e.* $V_j = V_i + \sigma_{ij}^2$

(*iii*) At merge points, the variance is computed along the longest (critical) path. In the case of two paths having the same length, the larger of the two-variance is chosen as the variance for that event.

$V_1 = 0$ $\quad V_5 = V_2 + \sigma_{2,5}^2 = 20/36$ $\quad V_9 = V_7 + \sigma_{7,9}^2 = 46/36^*$

$V_2 = (V_1 + \sigma_{12}^2) = 4/36$ $\quad V_6 = V_3 + \sigma_{3,6}^2 = 17/36$ $\quad V_{10} = V_9 + \sigma_{9,10}^2 = 50/36^*$

$V_3 = (V_1 + \sigma_{13}^2) = 1/36$ $\quad V_7 = V_3 + \sigma_{3,7}^2 = 13/36^*$

$V_4 = (V_1 + \sigma_{14}^2) = 4/36$ $\quad V_8 = V_6 + \sigma_{6,8}^2 = 33/36^*$

It is important to note that variances cannot be added as easily as is done, unless *two random variables are independent of each other.* In this case, it is assumed that two activities are independent of each other and hence their variances can be added.

The expected duration of the project is 19 days and the variance of this path is 50/36. If the exact probability distribution of the path is known, it would have been easy to find out the probability of completing the project in a given time. Since the variance of the path is known, the **Chebychev inequality**** could be used to get an estimate of probability for a given duration (if unimodality could be assumed, Camp-Neidel modification *to Chebychev's inequality will* give better estimate). *PERT* users, however, have used central limit theorem to claim that probability distribution of time for each event could be considered as normal. This is a strong assumption which greatly simplifies calculations and easily understood by most of the users of PERT. Assuming the normality, the probability of the project being completed by a certain date can be evaluated easily.

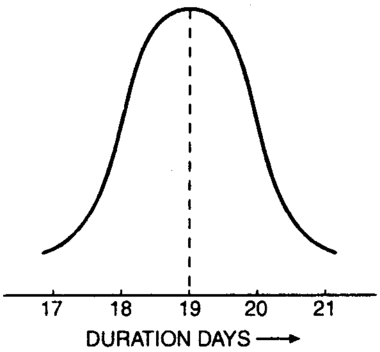


Fig. 25.48

Prob [*project duration 20 days*] = ?

$$\text{Prob } [D \le 20] = \text{Prob}\left[\frac{D-\mu}{\sigma} \le \frac{20-\mu}{\sigma}\right]$$

where μ is the mean of the distribution and σ is the standard deviation.

Now, the value $(D - \mu)/\sigma$ is a normalized value and usually written as Z. Then,

$$\textbf{Prob}\left[Z \le \frac{20-19}{\sqrt{(50/36)}}\right] = \textbf{Prob}\left[Z \le \frac{1}{7.07/6}\right] = \textbf{Prob } [Z \le 0.85] = 0.80$$ (from *Normal Distribution Table*).

The probability of finishing the job in less than or equal to 20 days is 0.80. The physical meaning of this statement is : If this job is done hundred times under same conditions, then there will be 80 occasions when this job will have taken 20 days or less to complete it. In other words, only 20 times, the job would have taken time longer than 20 days.

One of the main advantages of *PERT* approach to the management of a large scale project is in binding for contractual dates to finish the project. *A* designer would like to know the duration of the project that will have 95% chances of being completed. Let T_s be the scheduled duration such that

$$\textbf{Prob } [t \le T_s] = 0.95,\ \textbf{Prob}\left[\frac{t-\mu}{\sigma} \le \frac{T_s-\mu}{\sigma}\right] = 0.95,\ \textbf{Prob}\left[Z \le \frac{T_s-\mu}{\sigma}\right] = 0.95$$

From normal table $Z_{.95} = 1.64$. Therefore,

---

*Value represent the calculation along the longest (critical) path.

**The *Chebychev inequality* is stated as follows : Prob $[|\ x - \mu\ | > k\sigma] \le 1/k^2$.

That is, the probability of a random variable exceeding the mean value by k standard deviation is less than $1/k^2$. The *Camp-Neidel inequality* is Prob $[|\ x - \mu\ | > k\sigma] \le 1/(1.5k)^2$.

$$(T_s - \mu)/\sigma = 1.64, \text{ and } T_s = 19 + \frac{7.07}{6} \times 1.64 = 20.90 = 21 \text{ days.}$$

That is, if the designer says that he will need 21 days to complete this project, he has better than 95% chances of meeting the committed date. In larger projects, failure to meet the commited date could result in heavy penalties and great loss of good-will.

25.10-1 Illustrative Examples On PERT

Example 14. For *the project represented by the network diagram, find the earliest and latest times to reach each node, given the following data :*

Task	:	A	B	C	D	E	F	G	H	I	J	K
Least time	:	4	5	8	2	4	6	8	5	3	5	6
Greatest time	:	8	10	12	7	10	15	16	9	7	11	13
Most likely time	:	5	7	11	3	7	9	12	6	5	8	9

[VTU (BE Mech.) 2002]

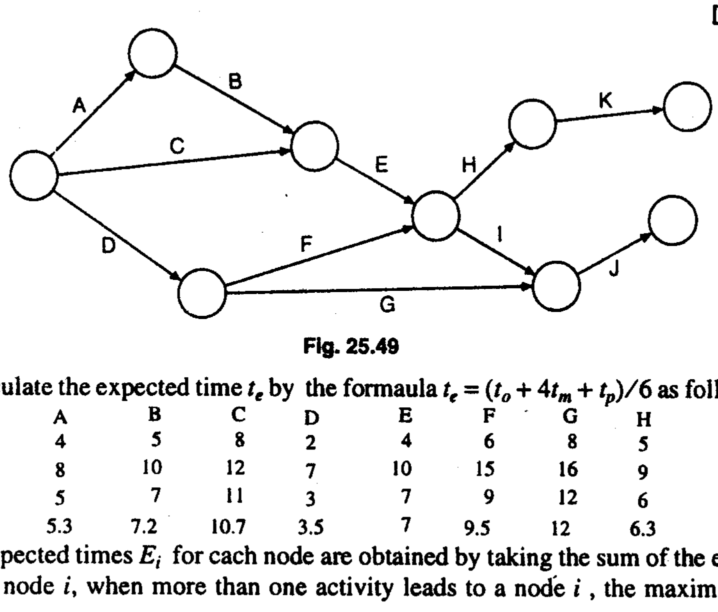


Fig. 25.49

Solution. First calculate the expected time t_e by the formaula $t_e = (t_o + 4t_m + t_p)/6$ as follows :

Task	:	A	B	C	D	E	F	G	H	I	J	K
t_o	:	4	5	8	2	4	6	8	5	3	5	6
t_p	:	8	10	12	7	10	15	16	9	7	11	13
t_m	:	5	7	11	3	7	9	12	6	5	8	9
t_e	:	5.3	7.2	10.7	3.5	7	9.5	12	6.3	5	8	9.1

Now, the earliest expected times E_i for each node are obtained by taking the sum of the expected times for all the activities leading to node i, when more than one activity leads to a node i , the maximum of E_i is selected. Therfore,

$E_1 = 0,\ E_2 = 0 + 5.3 = 5.3,\ E_3 = 0 + 3.5 = 3.5,\ E_4 = \max\ [5.3 + 7.2, 0 + 10.7] = 12.5,$

$E_5 = \max\ [12.5 + 7.0, 3.5 + 9.5] = 19.5,\ E_6 = 19.5 + 6.3 = 25.8,$

$E_7 = \max\ [19.5 + 5, 3.5 + 12] = 24.5,\ E_8 = 25.8 + 9.1 = 34.9,.\ E_9 = 24.5 + 8.0 = 32.5.$

To find the latest expected times to start with the latest time T_L for the last node as equal to E_i. Now moving backwards for each path, subtracting the expected time 't_e' for each activity link to have

$L_9 = 32.5, L_8 = 34.9, L_7 = 32.5 - 8 = 24.5, L_6 = 34.9 - 9.1 = 25.8,$

$L_5 = \min\ [25.8 - 6.3, 24.5 - 5] = 19.5,\ L_4 = 19.5 - 7 = 12.5$

$L_3 = \min\ [19.5 - 9.5, 24.5 - 12] = 10,\ L_2 = 12.5 - 7.2 = 5.3,$

$L_1 = \min\ [5.3 - 5.3, 12.5 - 10.7, 10 - 3.5] = 0.$

These calculations may be arranged in the following table :

Node	t_e	E_i	L_i	Slack
2	5.3	5.3	5.3	0
3	3.5	3.5	10.0	6.5
4	7.2	12.5	12.5	0
5	7.0	19.5	19.5	0
6	6.3	25.8	25.8	0
7	5.0	24.5	24.5	0
8	9.1	34.9	34.9	0
9	8.0	32.5	32.5	0

Example 15. *In Example 14, also find the critical path of the network :*

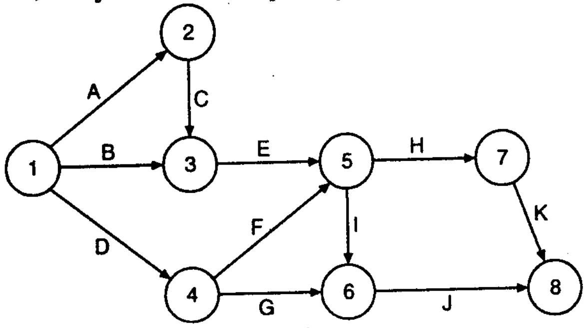


Fig. 25.50

Solution.

Expected Time Computations

Task	Last time *a*	Greatest time *b*	Most likely time *m*	Expected time $(a + b + 4m)/6$
A	4	8	5	5⅓
B	5	10	7	7⅙
C	8	12	11	10⅔
D	2	7	3	3½
E	4	10	7	7
F	6	15	9	9½
G	8	16	12	12
H	5	9	6	6⅓
I	3	7	5	5
J	5	11	8	8
K	6	13	9	9⅙

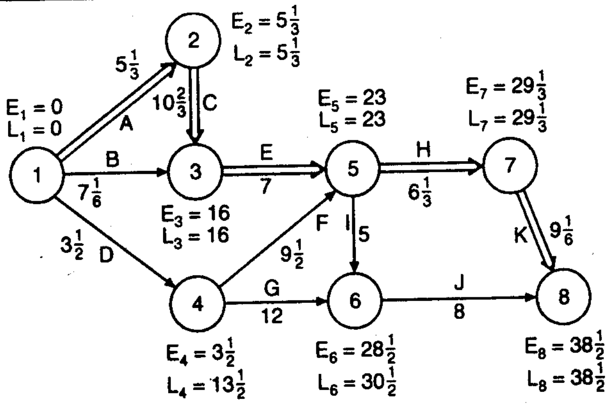


Fig. 25.51

Task	Expected time (t_e)	Start		Finish		Total float
		Earliest	Latest	Earliest	Latest	
A	5⅓	0	0	5⅓	5⅓	0
B	7⅙	0	8⅚	7⅙	16	8⅚
C	10⅔	5⅓	5⅓	16	16	0
D	3½	0	10	3½	13½	10
E	7	16	16	23	23	0
F	9½	3½	13½	13	23	10
G	12	3½	18½	15½	30½	15
H	6⅓	23	23	29⅓	29⅓	0
I	5	23	25½	28	30½	2½
J	8	28	30½	36	38½	2½
K	9⅙	29⅓	29⅓	31½	38½	0

Critical path is $A \to C \to E \to H \to K$.

Example 16. *A project has the following characteristics.*

Activity	Most Optimistic Time (*a*)	Most Pessimistic Time (*b*)	Most likely Time (*m*)
(1–2)	1	5	1.5
(2–3)	1	3	2
(2–4)	1	5	3
(3–5)	3	5	4
(4–5)	2	4	3
(4–6)	3	7	5
(5–7)	4	6	5
(6–7)	6	8	7
(7–8)	2	6	4
(7–9)	5	8	6
(8–10)	1	3	2
(9–10)	3	7	5

Construct a PERT network. Find critical path and variance for each event. Find the project duration at 95% probability.

Solution. Activity expected times and their variances are computed by the following formulae :

$$\textit{Expected time, } (t_e) = \frac{a+b+4m}{6}, \quad v = \left(\frac{b-a}{6}\right)^2$$

Activity	(*a*)	(*b*)	(4m)	t_e	v
(1–2)	1	5	6	2	4/9
(2–3)	1	3	8	2	1/9
(2–4)	1	5	12	3	4/9
(3–5)	3	5	16	4	1/9
(4–5)	2	4	12	3	1/9
(4–6)	3	7	20	5	4/9
(5–7)	4	6	20	5	1/9
(6–7)	6	8	28	7	1/9
(7–8)	2	6	16	4	4/9
(7–9)	5	8	24	6⅙	1/4
(8–10)	1	3	8	2	1/9
(9–10)	3	7	20	5	4/9

The network is constructed as below. the earliest and latest times of each event have been computed and indicated on the network. With the help of latest times, the longest path $1 \to 2 \to 4 \to 6 \to 7 \to 9 \to 10$ can be traced.

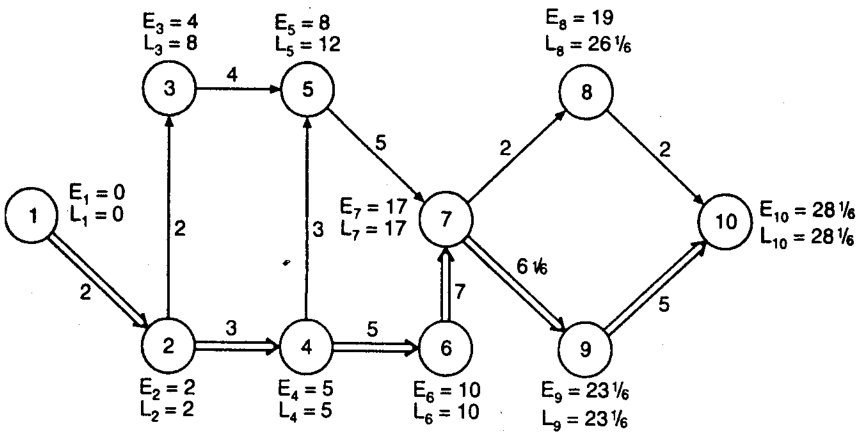


Fig. 25.52

Example 17. *Obtain the critical path and project duration for the following PERT network.*

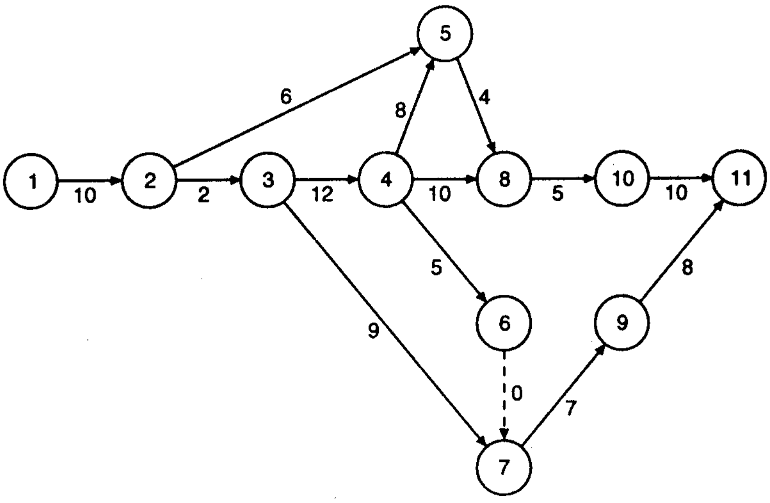


Fig. 25.53

Solution. *E*'s and *L*'s for all events are computed below on the network diagram by forward and backward passes respectively. This provides us the project duration as 51 days.

The network analysis table is construced as below.

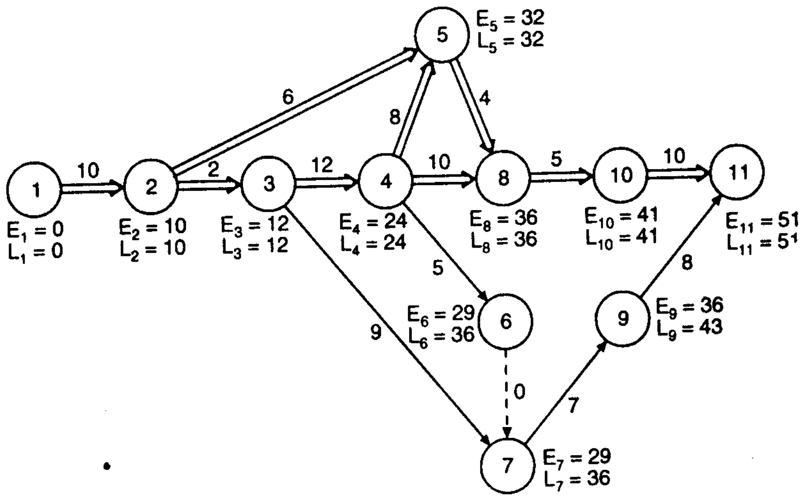


Fig. 25.54

Activity	Duration	Start		Finish		Total float
		Earliest	Latest	Earliest	Latest	
(1–2)	10	0	0	10	10	0
(2–3)	2	10	10	12	12	0
(2-5)	6	10	26	16	32	16
(3–4)	12	12	12	24	24	0
(3–7)	9	12	27	21	36	15
(4–5)	8	24	24	32	32	0
(4–6)	5	24	31	29	36	7
(4–8)	10	24	26	34	36	2
(5–8)	4	32	32	36	36	0
(6–7)	0	29	36	29	36	7
(7–9)	7	29	36	36	43	7
(8–10)	5	36	36	41	41	0
(9–11)	8	36	43	44	51	7
(10–11)	10	41	41	51	51	0

The critical path is traced along zero total float activities as 1 → 2 → 3 → 4 → 5 → 8 → 10 → 11 and is shown by double lines on the above network.

Example 18. *In the network shown below, the three time estimates for the activities are indicated. Number the events according to Fulkerson's rule and calculate the variance and expected time for each activity.*

Solution. Using the formula $t_e = (t_o + 4t_m + t_p)/6$, variance $\sigma^2 = (t_p - t_o)^2/36$, compute the following table.

Activity	t_o	t_m	t_p	t_e	σ^2
(1–2)	3	6	10	6.2	1.36
(1–3)	6	7	12	7.7	1.00
(1–4)	7	9	12	9.2	0.69
(2–3)	0	0	0	0.0	0.00
(2–5)	8	12	17	12.2	2.25
(3–6)	10	12	15	12.2	0.69
(4–7)	8	13	19	13.2	3.36
(5–8)	12	14	15	13.9	0.25
(6–7)	8	9	10	9.0	0.11
(6–9)	13	16	19	16.0	1.00
(8–9)	4	7	10	7.0	1.00
(7–10)	10	13	17	13.2	1.36
(9–11)	6	8	12	8.4	1.00
(10–11)	10	12	14	12.0	0.66

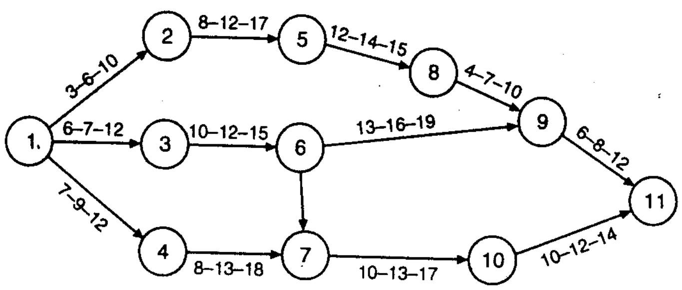


Fig. 25.55

Example 19. *A project is represented by the network shown below and has the following data :*

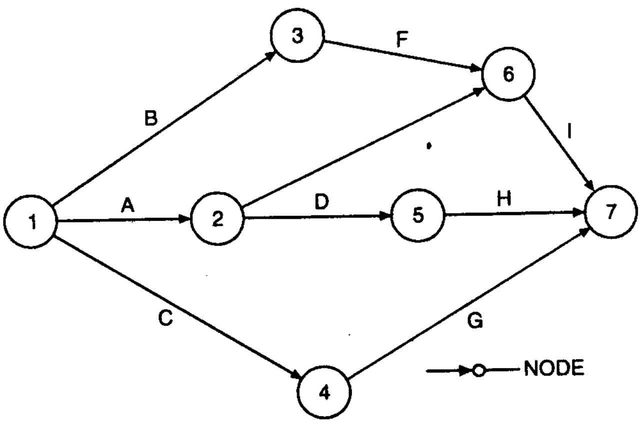


Fig. 25.56

Task	:	A	B	C	D	E	F	G	H	I
Leat time	:	5	18	26	16	15	6	7	7	3
Greatest time	:	10	22	40	20	25	12	12	9	5
Most likely time	:	8	20	33	18	20.	9	10	8	4

Determine the following :

(*i*) *expected task time and their variance,*

(*ii*) *the earliest and latest expected times to reach each node,*

(*iii*) *the critical path, and*

(*iv*) *the probability of node occuring at the proposed completion date if the original contract time of completing the project is 41.5 weeks*

Solution. (*i*) Proceeding as in above example obtain the following table :

Activity	t_o	t_p	t_m	t_e	σ^2
(1–2)	5	10	8	7.8	0.69
(1–3)	18	22	20	20.0	0.44
(1–4)	26	40	33	33.0	5.43
(2–5)	16	20	18	18.0	0.44
(2–6)	15	25	20	20.0	2.78
(3–6)	6	12	9	9.0	1.00
(4–7)	7	12	10	9.8	0.69
(5–7)	7	9	8	8.0	0.11
(6–7)	3	5	4	4.0	0.11

(*ii*) Proceeding exactly as in above *example*, find earliest times in usual notations.

$E_1 = 0, E_2 = 0 + 7.8,\ E_3 = 0 + 20 = 20, E_4 = 0 + 33 = 33,\ E_5 = 7.8 + 18 = 25.8,$

$E_6 = \max\ [7.8 + 20, 20 + 9] = 29,\ E_7 = \max\ [33 + 9.8, 25.8 + 8, 29 + 4] = 42.8.$

Moving backwards, calculate the latest times as before,

$L_7 = 42.8, L_6 = 42.8 - 4 = 38.8, L_5 = 42.8 - 8 = 34.3, L_4 = 42.8 - 9.8 = 33, L_3 = 38.8 - 9 = 29.8$

$L_2 = \min\ [34.8 - 18, 38.8 - 20] = 16.8,\ L_1 = \min\ [16.8 - 7.8, 29.8 - 20, 33 - 33] = 0.$

(*iii*) To find the critical path, calculate slack time by taking difference between the earliest expected times and latest allowable times. Calculations are given in the following table and critical path is shown by double line in the following figure.

Node (i)	t_e	E_i	L_i	Slack	Var. σ_i
2	7.8	7.8	16.8	9.0	0.69
3	20.0	20.0	29.8	9.8	0.44
4	33.0	33.0	33.0	0.0	5.42
5	18.0	25.8	34.8	9.0	1.13
6	9.0	29.0	38.8	9.8	1.44
7	9.8	42.8	42.8	0.0	6.12

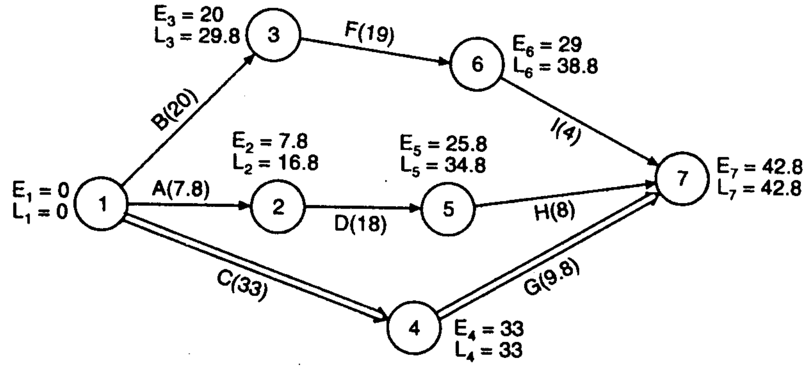


Fig. 25.57.

(*iv*) The scheduled time of completing the project is 41.5 weeks. Therefore, the distance in standard deviations, that schedule time from earliest expected times E_i, is given by

$$D_i = \frac{ST_i - E_i}{\sqrt{[Var.(i)]}} = \frac{41.5 - 42.8}{\sqrt{(6.12)}} = -0.52$$

where ST_i denotes the schedule time.

Therefore, $P(Z \geq -0.52) = 1 - P[Z \leq 0.52] = 1 - 0.70 = 0.30$ (from *Normal Table*) which is the area under the standard normal curve bounded by ordinates at $x = 0$, and $x = 0.52$.

From this it is concluded that if the project is performed 100 times under the same conditions, there will be 30 chances when this job would take 41.5 weeks or less to complete it.

Example 20. *The following table lists the jobs of a network with their estimates.*

Job $(i–j)$	Duration (days)		
	Optimistic (t_o)	Most likely (t_m)	Pessimistic (t_p)
(1–2)	3	6	15
(1–6)	2	5	14
(2–3)	6	12	30
(2–4)	2	5	8
(3–5)	5	11	17
(4–5)	3	6	15
(6–7)	3	9	27
(5–8)	1	4	7
(7–8)	4	19	28

(*i*) ***Draw the project network,*** (*ii*) ***calculate the length and variance of the critical path, and***
(*iii*) ***what is the approximate probability that the jobs on the critical path will be completed in 41 days.***

Solution. Using the formula $t_e = (t_o + 4t_m + t_p)/6$ and $\sigma^2 = (t_p - t_o)^2/36$, calculate t_e and σ^2.

Activity	:	(1–2)	(1–6)	(2–3)	(2–4)	(3–5)	(4–5)	(6–7)	(5–8)	(7–8)
t_e	:	7	6	14	5	11	7	11	4	18
σ^2	:	4	4	16	1	4	4	16	1	16

The earliest expected times are calculated as

$E_1 = 0, E_2 = 0 + 7 = 7, E_3 = 7 + 14 = 21, E_4 = 7 + 5 = 12, E_5 = \max\ [21 + 11,\ 12 + 7] = 32,$

$E_6 = 0 + 6 = 6,\ E_7 = 6 + 11 = 17, E_8 = \max\ [32 + 4,\ 17 + 18] = 36.$

The latest expected times are calculated as :

$L_8 = 36, L_7 = 36 - 18 = 18, L_6 = 18 - 11 = 7,\ L_5 = 36 - 4 = 32, L_4 = 32 - 7 = 25,\ L_3 = 32 - 11 = 21,$

$L_2 = \min\ [21 - 14, 25 - 5] = 7, L_1 = \min\ [7 - 7, 7 - 6] = 0.$

For the critical path, calculate the slack time by taking the difference between the earliest expected times and latest allowable times. Critical path is shown by double line in the following diagram.

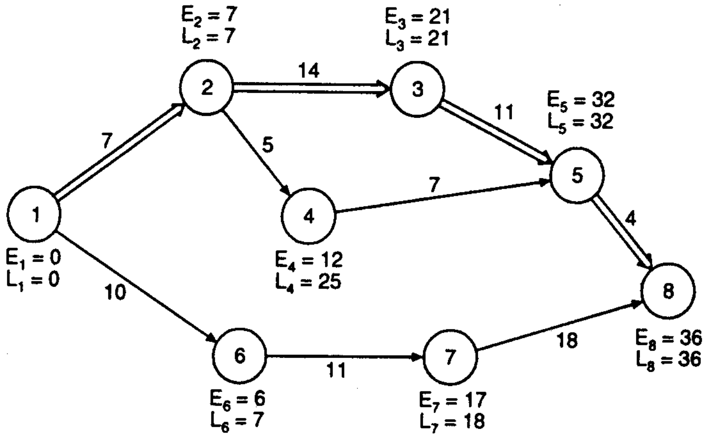


Fig. 25.58

The optimum length of the critical path is 36 days and variance of the critical path is $4 + 16 + 4 + 1 = 25$.

Now the scheduled time of completing the jobs is given 41 days. Therefore, the distance in standard deviations, that schedule time from earliest expected time is given by

$$D_i = \frac{St_i - E_i}{\sqrt{[Var.(i)]}} = \frac{41 - 36}{\sqrt{25}} = \frac{5}{5} = 1.$$

where ST_i denotes the scheduled time.

Hence $P(Z \le D_i) = 0.84$, which is area under standard normal curve bounded by the ordinates $x = 0$ and $x = 1$.

This concludes that only 16 times the job would take time longer than 41 days.

Example 21. *Consider the network shown in the figure given below. The estimates of* t_o, t_m and t_p *are shown in this order for each of the activities on the top of the arcs denoting the respective activities. Find the probability of completing the project in 25 days.*

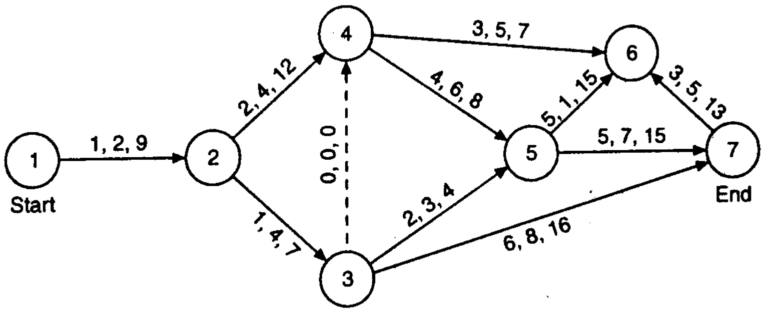


Fig. 25.59

Solution. Using the formula for expected activity duration t_e and the variance σ^2, these values are obtained as shown below :

Activity	t_o	t_m	t_p	t_e	σ^2
(1–2)	1	2	9	3	1.78
(2–3)	1	4	7	4	1.00
(2–4)	2	4	12	5	2.78
(3–4)	0	0	0	0	0.00
(3–5)	2	3	4	3	0.11
(3–7)	6	8	16	9	2.78
(4–5)	4	6	8	6	0.44
(4–6)	3	5	7	5	0.45
(5–6)	½	1	3½	1	0.03
(5–7)	5	7	15	8	2.78
(6–7)	3	5	13	6	2.78

Now calculate the earliest expected times and the latest allowable time making use of all t_e's. For critical path, determine the slack time by computing the difference between the earliest expected times and latest allowable times. The critical path is shown by double line in the following figure.

The optimum length of critical path is 22 days and the variance of the critical path is 25.78.

But, the scheduled time of completing all the activities is 25 days. Therefore,

$$D_i = \frac{ST_i - E_i}{\sqrt{[Var.(i)]}}, = \frac{25-22}{\sqrt{(7.78)}} = 1.08. \text{ where } ST_i \text{ denotes schedule time.}$$

From standard normal tables, $P(Z \leq D_i) = 0.86$

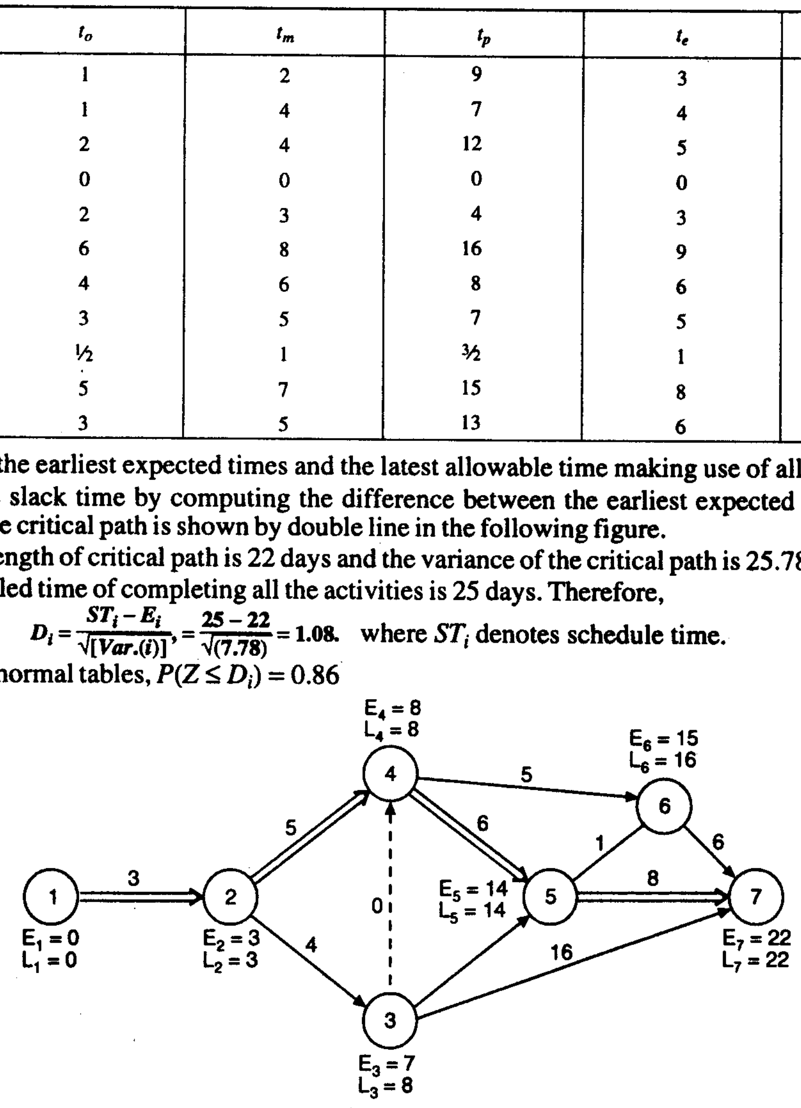


Fig. 25.60

Example 22. *In the following table optimistic, most-likely and pessimistic times are respectively shown against each connected activity from 10 to 100 in a project. Find the critical path by constructing a network.*

The scheduled completion time for the project is 48 days. Calculate the probability of finishing the project within this time (given that 89.5% probability corresponds to a normal deviation of + 1.25).

Activity	Times	Activity	Times
(10–20)	4, 8, 12	20–30	1, 4, 7
(20–40)	8, 12, 16	30–50	3, 5, 7
(40–50)	0, 0, 0	40–60	3, 6, 9
(50–70)	3, 6, 9	50–80	4, 8, 6
(60–100)	4, 6, 8	70–90	4, 8, 12
(80–90)	2, 5, 8	90–100	4, 10, 16

Solution. The expected activity time t_e and the variance σ^2 are calculated as given in the following table.

Activity	t_o	t_m	t_p	$t_e = (t_o + 4t_m + t_p)/6$	$\sigma^2 = [(t_p - t_o)/6]^2$
(10–20)	4	8	12	8	1.78
(20–40)	8	12	16	12	1.78
(40–50)	0	0	0	0	0.00
(50–70)	3	6	9	6	1.00
(60–100)	4	6	8	6	0.44
(80–90)	2	5	8	5	1.00
(20–30)	1	4	7	4	1.00
(30–50)	3	5	7	5	0.44
(40–60)	3	6	9	6	1.00
(50–80)	4	8	6	7	0.11
(70–90)	4	8	12	8	1.78
(90–100)	4	10	16	10	4.00

To find the *critical path*, calculate the earliest expected time and the latest allowable time and then obtain the difference between these two. Variances for each node (i) are calculated by rules given after ***Table* 25.5.** This information is given in the following table.

Node	E_i (T_e)	L_i (T_L)	Slack T_s	Var. (i)
20	8	8	0	1.78
30	12	15	3	2.78
40	20	20	0	3.56
50	20	20	0	3.22
60	26	36	10	4.56
70	26	26	0	4.22
80	27	29	2	3.33
90	34	34	0	6.00
100	44	44	0	10.00

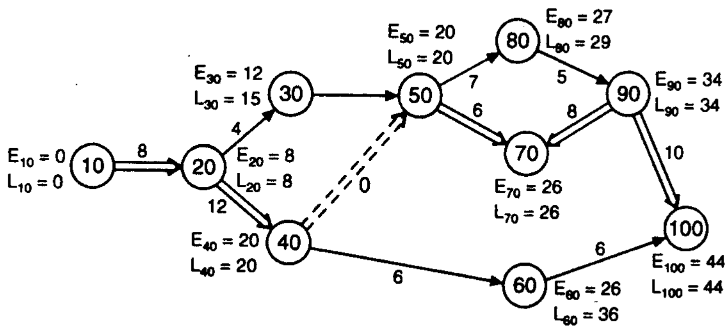


Fig. 25.61

The critical path is shown in the following figure by double line.

But, the scheduled completion time for the project is 48 days, therefore

$$D_i = \frac{48 - 44}{\sqrt{10}} = 1.27.$$

Hence $P(Z \le D_i) = 0.894$, the area under the standard normal curve bounded by the ordinates $x = 0$ and $x = 1.27$.

Example 23. *Assuming that the expected times are normally distributed, find the probability of meeting the schedule date as given for the network.*

Activity ($i - j$)	Days		
	Optimistic (a)	Most likely (m)	Pessimistic (b)
(1–2)	2	5	14
(1–3)	9	12	15
(2–4)	5	14	17
(3–4)	2	5	8
(4–5)	6	6	12
(3–5)	8	17	20

Scheduled project completion date is 30 days. Also find the date on which the project manager can complete the project with a probability of 0·90.

Solution. The expected activity time t_e and the activity variance σ^2 are calculated in the following table.

Activity	(*a*)	(*m*)	(*b*)	t_e	σ^2
(1–2)	2	5	14	6	4
(1–3)	9	12	15	12	1
(2–4)	5	14	17	13	4
(3–4)	2	5	8	5	1
(4–5)	6	6	12	7	1
(3–5)	8	17	20	16	4

For the critical path, determine the earliest expected time and the latest allowable time and then find the difference between the two.

The critical path is shown in the following figure by double line. The optimum length of the critical path $1 \to 2 \to 4 \to 5$ or $1 \to 3 \to 5$ is 28 days and the variance of the critical path is 9 or 5.

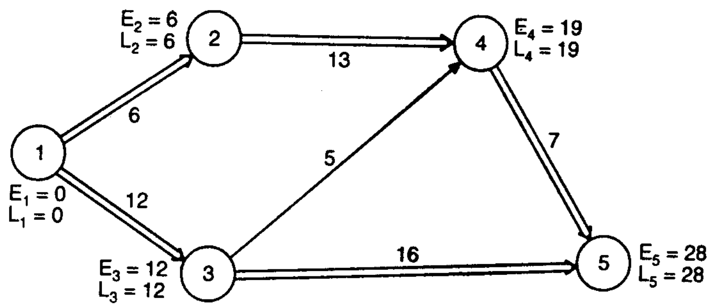


Fig. 25.62

But, the scheduled time of completing all the activities is given 30 days. Therefore, the distance in standard deviation, that schedule time from earliest **expected time**, is given by

$$D_i = \frac{ST_i - E_i}{\sqrt{[Var\ (i)]}}, \text{ where } ST_i \text{ denotes schedule time}$$

$$= \frac{30 - 28}{\sqrt{9}} \text{ or } \frac{30 - 28}{\sqrt{5}} = 0{\cdot}65 \text{ or } 0{\cdot}89$$

Hence $P(Z \le D_i) = 0{\cdot}71$ or 0.81 [from *standard normal tables*].

25.11. UPDATING

Many benefits of network planning may be lost if updating or review of the network from time to time is not carried out. The updating implies the study of the progress that has been made and its impact on the remaining jobs or activities. This also includes revising estimates of time on various activities, re-drawing of the remaining network and calculation of critical paths. Latter two steps are necessary because original values were the estimates, and the critical path is based on these estimates. If these estimates have changed or better values are found, it is not necessary that the new critical path is the same as before.

Consider a situation that on the 10th day since the beginning of work, the designer finds that activities 2–5, 7–9 and 6–8 are completed and remaining ones are due to start tomorrow. The designer is interested to know the chances of completing this project by 20th day.

For this, only the remaining portion of the network is relevant. Activities that are completed by now cannot influence the remaining network. The revised network (consisting of incomplete jobs) is :

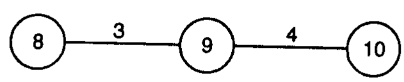


Fig. 25.63

The expected duration for the remainder of the project is 7 days and the total variance is 8/36. In order to finish the job by 20th day, there are 8 more days available.

Therefore, *Prob. of finishing the job in 8 days* is given by

$$\text{Prob}\,[t \le 8] = \text{Prob}\left[Z \le \frac{8-7}{\sqrt{8/36}}\right] = \text{Prob}\,[Z \le 2.12] = 0.983 \qquad \text{(from } \textit{normal tables}\text{).}$$

This is much higher than the original value of 83% calculated before the project was commenced. In this example, it is assumed that original estimates for activities 8–9 and 9–10 are still valid. If they are not the same, revised values are used.

In order to derive maximum benefits from the network technique, updating must be done as frequently as economically possible. Revising the network, recalculating the critical path, and finding new estimates will involve management and personnel time, but to maintain the dynamic nature of the network, updating is a necessary evil.

25.11-1. An Illustrative Example

To explain the method of up-dating, we consider the following network.

Now we suppose that the process is reviewed at the end of 10th day and it is found that—

(*i*) activities (0—1), (0—2) and (1—3) are completed,

(*ii*) activity (2—3) is in progress and will take 6 days more,

(*iii*) activity (2—4) is in progress and will take 7 days more,

(*iv*) also, it is estimated that on account of arrival of new machine, the activity (3—5) will take 6 days only.

Above information can be put into a tabular form as shown below :

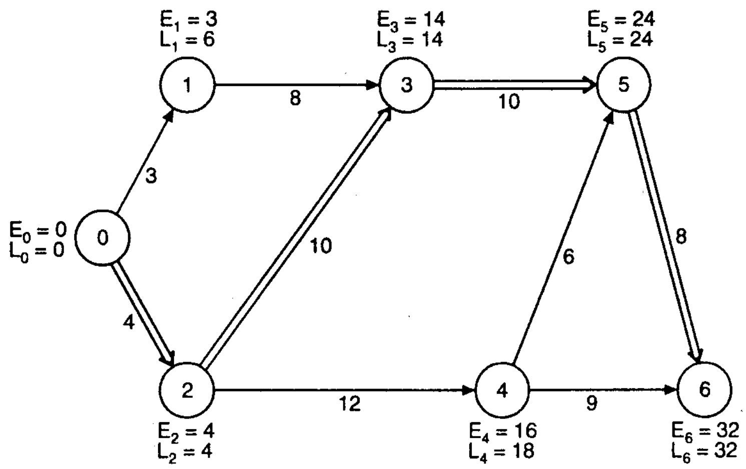


Fig. 25.64

Review Time After 10 Days

Activity	More than required (days)	Situation
(0–1)	0	Complete
(0–2)	0	Complete
(1–3)	0	Complete
(2–3)	6	In Progress
(2–40	7	In Progress
(3–5)	6	Not Started
(4–5)	6	Not Started
(4–6)	9	Not Started
(5–6)	8	Not Started

The up-dating can be performed by two methods. The first method is to use the revised time estimates and calculate from the initial starting event. Secondly, the more convenient method is to change the complete work to zero duration and bunch all the jobs already done into a single arrow called the *elapsed time arrow*. But, the nodes in the revised network are numbered in a *different* manner. The time duration assigned to the activities are the revised times. In the revised network shown in Fig 22.64, activity (0—20) indicates the elapsed time of 10 days. Activities (20—30) and (20—40) are allotted the times which are required for their completion. Along other activities we put their revised time estimates, Now, after computing the *earliest expected* times and *latest* permissible times, we observe that *critical path* is changed to $0 \rightarrow 20 \rightarrow 40 \rightarrow 50 \rightarrow 60$.

The total duration is also reduced now by one day.

To represent the information regarding the original schedule, bar charts (Fig.25.65) can be used in the process of updating.

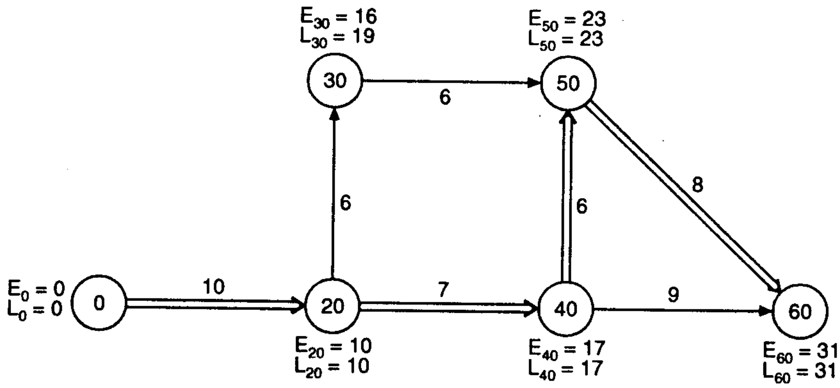


Fig. 25.65

The nature of the project at the end of 10 days is represented by shading the bars as shown in the Fig. 25.66. The updating line indicates that activities 1–3, 2—3 and 2—4 are in progress, but after reviewing the project we observe that the activity 1—3 has already been completed. So we shade this duration through the total length. The changes in the lengths of bars to show the increase or decrease in activity duration is represented with the help of dotted lines.

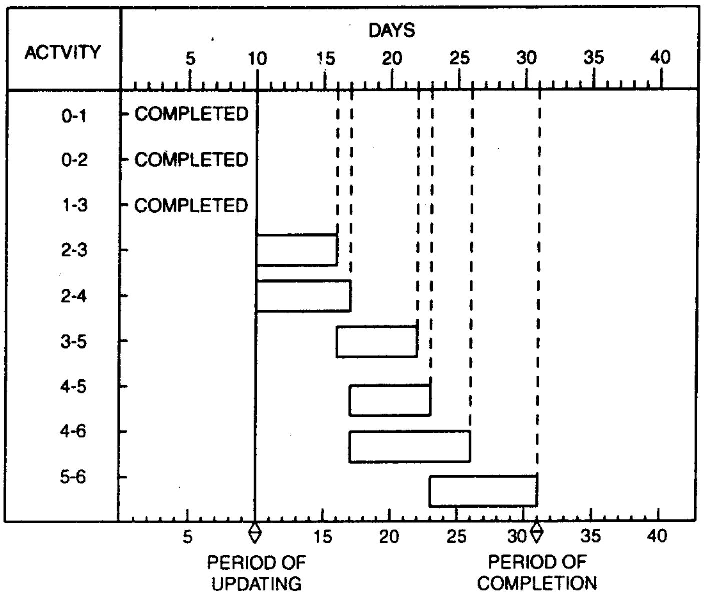


Fig. 25.66

For example, activity 2—3 needs 6 more days and is therefore extended upto 16, and activity 2—4 is extended upto 17. Activity 3—5 can be completed in 6 days only in comparison to the original 10 days, but it can be started only when activity 1—3 becomes complete, that is, after 6 days. So the bar needs shifting to the right and is cut to the proper length of 6 days. Now activity 4—5 can start after 17 days only and is therefore shifted to the right by one day. In the like manner, activity 5—6 can be started after 23 days and is shifted by one day to the left. The bar chart corresponding to the revised network will now look as shown in *Fig.* 25.66.

There is no special to decide about the frequency of up-dating, *i.e.* how many times the up-dating should be done. This depends upon the size and nature of the project and also upon the attitude of the management. But, however, a general opinion is that the frequency of up-dating may be less in the beginning but should be more frequent near the completion of the project. Of course, some slippages in the beginning can be absorbed, but a slip near the completion of the project will delay the project. In small projects, as the time of absorbing the slippages becomes less, more frequent up-dating is called for.

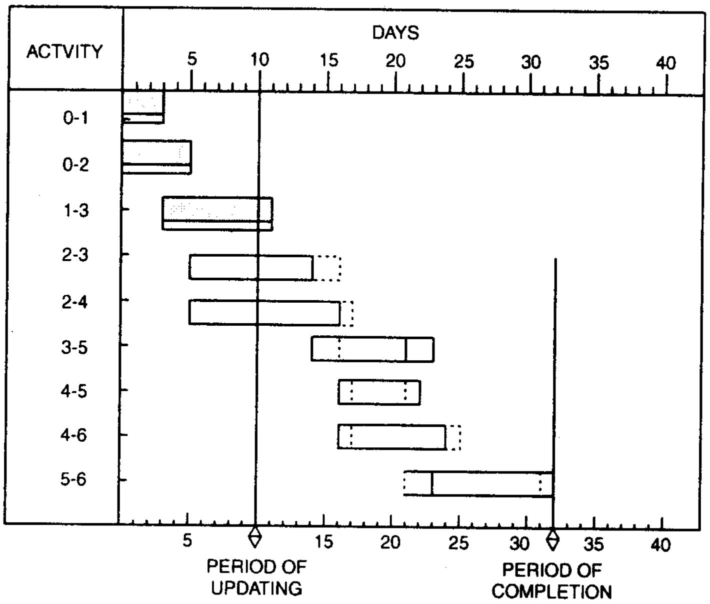


Fig. 25.67

25.12. RESOURCE ALLOCATION

While developing the PERT and CPM networks we have generally assumed that sufficient resources are available to perform the various activities. In every production enterprise, resources are always limited and the management always wants to assign these various activities in such a manner that there is best possible utilization of available resources. At a certain time the demand on a particular resource is the cumulative demand of that resource on all the activities being performed at that time. Proceeding according to the developed plan, the demand on a certain type of resource may fluctuate form very high at one time to a very low at the other. If it is a material or unskilled labour which has to be procured from time to time, the fluctuation in demand will not much affect the cost of the project. But, if it is some personnel who cannot be hired and fired during the project or machines which are to be hired for the entire duration of the project, the fluctuation in their demand will affect the cost of the total project due to high idle times. In order to reduce the idle time, the activities on non-critical paths are shifted by making use of the floats and an alternative schedule is generated comparing the more important resources with the object of smoothening the demand on resources. *PERT* and *CPM* techniques provide us valuable guidelines for most systematic and economic allocation of resources. The resources are some sort of variables : labour, capital and equipment.

In the preceding sections, it has already been explained that the presence of *slack* or *float* for any event or activity enables the production manager to delay that activity for some time and utilise these resources to some more urgent activities. The main object of the organization is not to waste the man-hours or materials. The specified quantity of resources should be available at the desired moment so that the work is not held-up and the plant, equipment and the space should be utilized to its maximum possible capacity.

The resource allocation procedure consists of two main acitvities : *resource smoothing* and *resource levelling,* depending upon the type of constraints.

Resource Smoothing. If the constraint is the total project duration, then the resource allocation only smoothens the demand on resources in order that the demand of any resource is as uniform as possible. The periods of maximum demand for resources are located and the activities according to their float values are shifted for balancing the availability and requirement of resources. So the intelligent utilization of floats can smoothen the demand of resources to the maximum possible extent. Such type of resource allocation is called *'Resource 'Smoothing' or Load Smoothing'*.

Resource Levelling. There are various activities in a project demanding varying levels of resources. The demand on certain specified resources should not go beyond the prescribed level. This operation of resource allocation is called *'Resorce Levelling'* or *'Load Levelling'*.

Althouth the overall resources of the organization are limited, but these should not go beyond the maximum amount required to perform an activity among all the activities in the process, otherwise that particular activity cannot be completed. In the process of resource levelling, whenever the availability of a resource becomes less than its maximum requirement, the only alternative is to delay the activity having large large float. In case, two or more activities require the same resources, the activity with minimum duration is chosen for resource allocation,

25.12-1. Main Steps in Resource Smoothing

Step 1. The first step in resource smoothing is to determine the maximum requirement. One way is to draw the time scaled version of the network and assign the resource requirements to activities.

Step 2. Then, below the time scaled network, the cumulative resource requirements for each time unit are plotted.

Step 3. The resource histogram is plotted on the basis of early start times or the late start times of the activities. These resource histograms establish the framework under which the smoothing or levelling must occur.

25.12-2 An Illustrative Example

To illustrate the resource smoothing operation, we consider the network shown in Fig. 25.68 (see page 367). For simplicity, only kind of resources, i.e., crew size, has been considered.

The manpower required for each activity is given below.

Activity	Crew Size (Men)
(0–1)	4
(1–2)	3
(1–3)	3
(2–4)	5
(3–5)	3
(3–6)	4
(4–7)	3
(5–7)	6
(6–8)	2
(7–9)	2
(8–9)	9

Step 1. The earliest and latest times of events are computed and indicated along the nodes in the above network diagram. The critical path is shown as $0 \to 1 \to 3 \to 5 \to 7 \to 9$, where the total project duration is 20 weeks.

Step 2. In the time scaled version of the network (which is also called *squared network*), first of all the critical path is drawn along a straight line. Then the non-critical paths are added as shown in *Fig.* 25.68.

The resource requirements are indicated along the arrows.

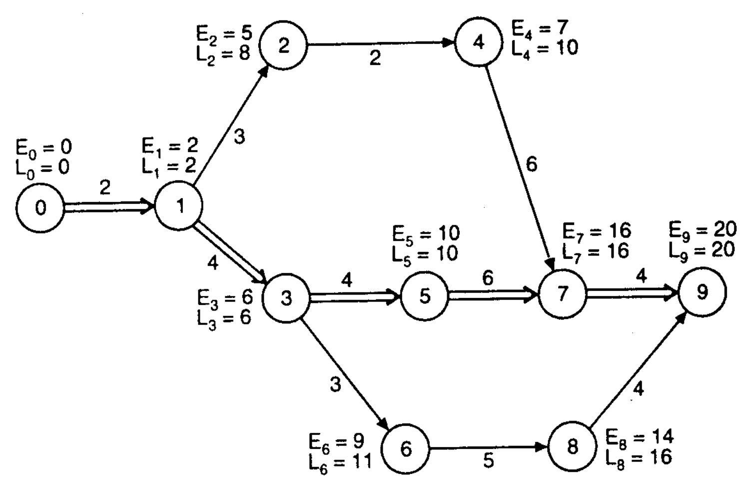


Fig. 25.68

Step 3. After the squared network, draw the resource histogram as shown below in *Fig*. 25.69. This is based on the earliest start times, and is obtained by vertically summing up the man-power requirements for each week.

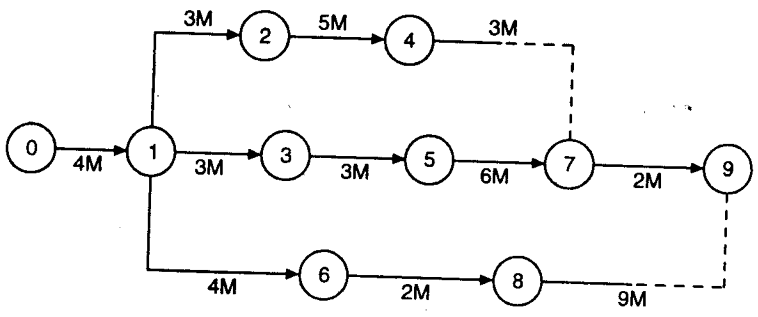


Fig. 25.69

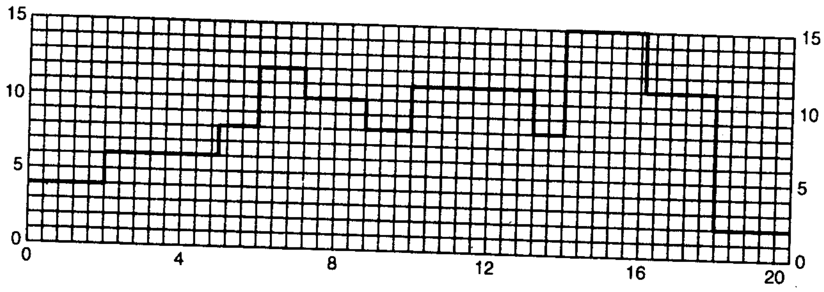


Fig. 25.70

Here we observe that the maximum demand of 15 men occurs in 15th and 16th week.

Step 4. ***(Shifting of Activities).*** To smoothen the resources (load) the activities will have to be shifted depending upon the floats. Path 3 → 6 → 8 → 9 has a float of two weeks, and the activities 6—8 and 8—9 are shifted to the right in order that the starting of each is delayed by two weeks. Similarly, activity 4—7 can be shifted to the right so that it starts on 10th day instead of starting on 7th day. After making the necessary shifting, the network is drawn as shown in the *Fig*.25.70.

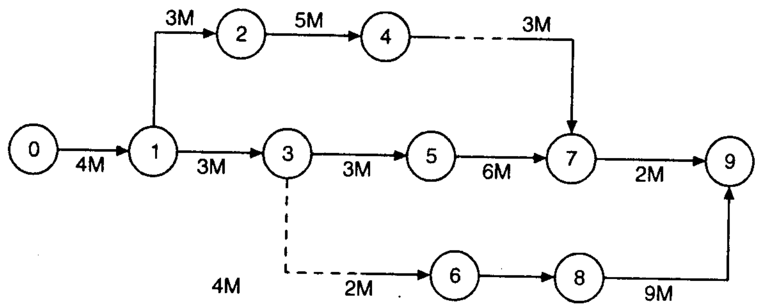


Fig. 25.71

Step 5. The resource histogram for above network (*Fig*. 25.71) is drawn in *Fig* 25.72, which indicates that maximum manpower required is 11 men. Hence with new schedule, the same project can be accomplished in the same duration of 20 weeks by 11 men as compared to 15 men for the previous schedule.

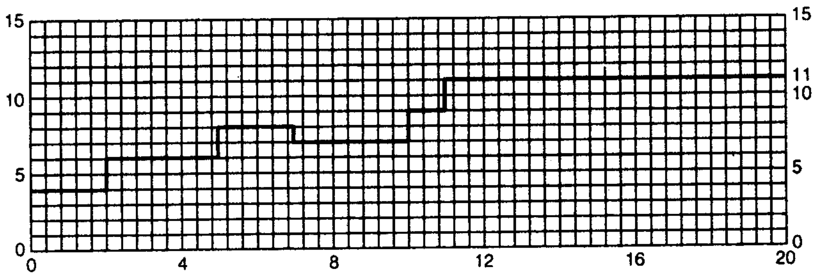


Fig. 25.72

25.12-3 Resource Levelling

Resource levelling is done if the restriction is on the availability of manpower. Suppose only 9 men are available for the execution of the project. Since the demand cannot be reduced to 9 by smoothing, a new

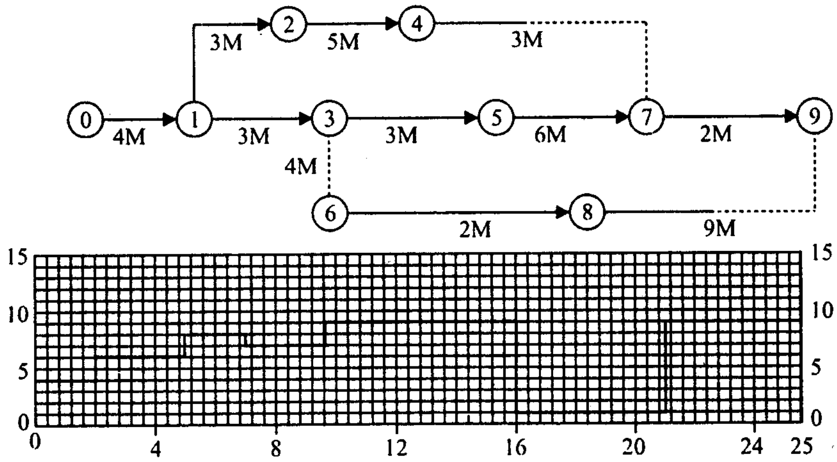


Fig. 25.73

scheduling situation will occur. In order to bring down the peak to 9, the only alternative is to extend the project duration as shown in the *Fig*. 25.73. In this network, it is worthnoting that there is no critcal path and the project duration is increased to 25 weeks.

If the number of categories of resources considered becomes more than one, then a compromise will have to be made to the minimum level of each resource in the resource smoothing.

25.12-4. Illustrative Example

Example 24. *Given the following information, suggest some appropriate allocation schedule :*

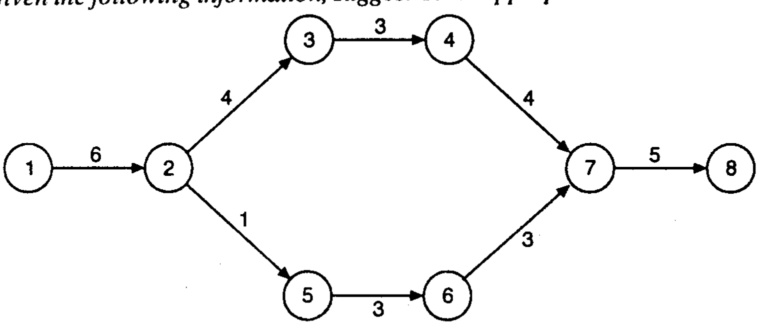


Fig. 25.74

and the Resource Table is as given below :

Critical Activity			Non-Critical Activity		
Activity	Men/day	Men	Activity	Men/day	Men
1–2	48	8	2–5	2	2
2–3	16	4	5–6	9	3
3–4	18	6	6–7	12	4
4–7	16	4			
7–8	20	4			

Solution. The maximum duration of the project is 22 days and the maximum number of laboures required is 8 for the activity 1–2. The earliest time to start various activities are calculated and then a time-scale network is constructed by taking the critical activities on the horizontal line and slack activities above it (see *Fig*.25.74).

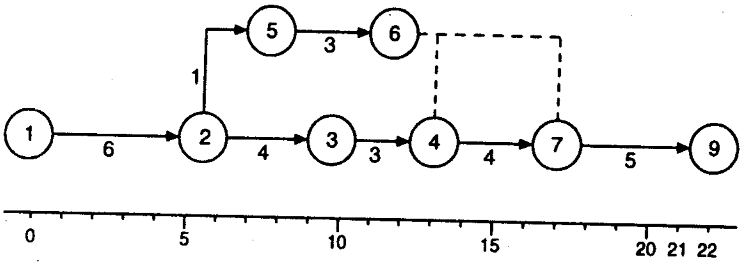


Fig. 25.75

Here for the activity 1—2 we require 8 men for 6 days, for the activity 2—3 we require 4 men for 4 days. But, the acitivities 2—5 and 5—6 can also be run simultaneously with the activity 2—3 because the activity 2—5 requires 2 men for one day and activity 5—6 requires 3 men for 3 days. So the requirement of man during first 10 days will be as follows :

Activity	Men	Days
1–2	8	6
2–3 and 2–5	4 + 2	1 } 4 days
2–3 and 5–6	4 + 3	3

Hence 8 men can easily do the work for first 10 days.

Now the activity 3—4 requires 6 men and if the activity 6—7 is also to run simultaneously, then we need only 10 men for first 3 days which is not at all possible. But, there being a float of 4 days, with activity 6—7, it can be delayed by 3 days, *i.e.* it can be started after 13 days and run together with activity 4—7. Hence, with this resource allocation plan, the whole project can be finished by assigning 8 men.

The labour allocation can be easily represented by the following histogram.

Days Number :	1—6	7	8—10	11–13	14—16	17	18—22
Men Busy :	8	7	8	6	8	4	4

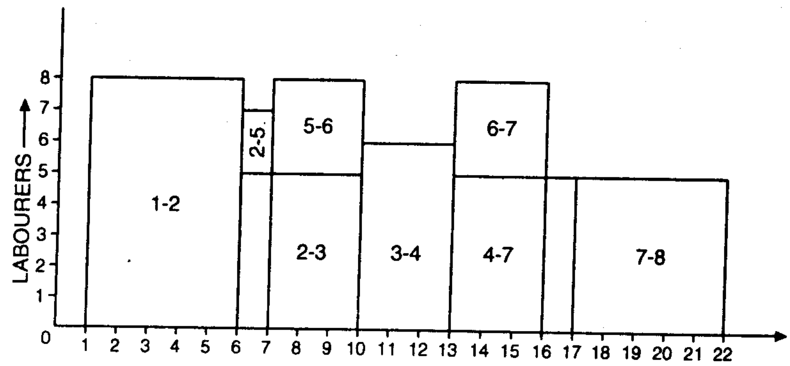


Fig. 25.76

25.13. USES OF PERT/CPM (NETWORKS) FOR MANAGEMENT

(1) The PERT/CPM techniques help the management in properly planning the complicated projects, controlling working plan and also keeping the plan upto-date. These are also helpful in searching the potential spots and in taking corrective measures.

(2) The network techniques provide a number of checks and safeguards against going astray in developing the plan for the project. Thus there are little chances of over-sight of certain activities and events.

(3) These techniques help the management in reaching the goal with minimum time and least cost and also in fore-casting the probable project duration and the associated cost.

(4) The networks clearly designate the responsibilities of different supervisors. The supervisor of an activity himself knows the time schedule precisely and also the supervisors of other activities with whom he has to co-operate.

(5) The flexibility of the network permits the management to make the necessary alterations and improvements as and when they are needed. These allocations can be made during the deployment of resources or reviewing.

(6) Application of network techniques has resulted in better managerial control, better utilization of resources, improved communication and progress reporting, and better decision making.

(7) Application of PERT/CPM techniques have resulted in saving of time which directly results in saving of cost. Also, saving in time or early completion of the project results in earlier return of revenue and introduction of the product or process ahead of the competitors, resulting in increased profits.

25.14. APPLICATION AREAS OF PERT/CPM TECHNIQUES

Though the list containing the PERT/CPM application areas is very large, but these techniques are very widely used in the following typical areas.

(1) **Building Construction.** It is one of the largest areas in which the network techniques of project management have found its applications.

(2) **Administration.** Networks are used by the administration for streamlining paperwork system, for making major administrative changes in the system, for long range planning and for developing staffing plans, etc.

(3) **Manufacturing.** The design development, and testing of new machines, installing machines and plant layouts are a few examples of its applications to the manufacturing function of a firm.

(4) **Maintenance Planning.** Maintenance and shutdown of power plants, chemical plants, steel furnaces and overhauling of large machines can be carried out by using *PERT* techniques.

(5) **Rssearch & Development.** The research and development is the most extensive area where *PERT* techniques are used for development of new products, processes and systems.

(6) **Inventory Planning.** Installation of production and inventory control, acquisition of spare parts, etc, have been greatly helped by network techniques.

(7) **Marketing.** Networks are also used for advertising programmes for development and launching of new products and for planning their distribution.

25.15. DISADVANTAGES OF NETWORK TECHNIQUES

Besides several advantages, the following difficulties are faced by the management while using the network techniques :

(1) The difficulty arises while securing the realistic time estimates. In the case of new and non-repetitive type of projects, the time estimates produced are often mere guesses.

(2) It is also sometimes troublesome to develop a clear logical network. This depends upon the data input and thus the plan can be no better than the personnel who provides the data..

(3) The natural tendency to oppose changed results in the difficulty of persuading the management to accept these techniques.

(4) Determination to the level of network detail is another troublesome area. The level of detail varies from planner to planner and depends upon the judgement and experience,

(5) The planning and implementation of networks require personnel trained in the network methodology. Managements are reluctant to spare the existing staff to learn these techniques or to recruit trained personnel.

SELF-EXAMINATION QUESTIONS

1. Explain in brief : PERT, CPM, crashing, dummy activities and lead time with reference to project management. **[Meerut 2002]**
2. Give in brief the role of statistical analysis in project management.
3. Explain the following terms in the context for project management.
(*i*) Resource float, (*ii*) Activity variance, (*iii*) Project variance.
4. How uncertainty can be incorporated in *PERT* model ?
5. Discuss in brief the following concepts in *PERT/CPM.*
(*i*) Time-cost trade-off, (*ii*) Resource levelling in a project.
6. What is critical path analysis ? Describe with illustration its utility in project planning and control.
7. How can the probabilistic network provide data helpful for managerial decisions.
8. Define "Critical path", "Slack time", "Resource levelling" and "Dummy activity" with reference to *PERT* and *CPM.*
9. (*a*) In PERT, the approach is probabilistic. Explain.
(*b*) Which is the type of theoretical distribution used in the determination of expected time in *PERT*? Give the equations for the expected time and variance.
10. What do you mean by *PERT*–cost ? Explain its utility.
11. (*a*) What is the difference between the time estimates of a *PERT* activity and a *CPM* activity ? **[Meerut 2002]**
(*b*) Distinguish between an activity and an event.
(*c*) State Fulkerson's rule for numbering the nodes in a network.
12. Explain with suitable examples, the following tems :
(a) Simplified *PERT* (b) Full *PERT* (c) *CPM*
13. (a) What is network analysis ? When is it used ? What is meant by the phrase 'critical path' ? When is any sequence of activities in a network critical ? Why should we want to know which activities are critical and which are not ?
(b) Explain the following terms in *PERT/CPM :*
(i) Earliest time, (ii) Latest time,(iii) Total activity slack, (iv) Event slack, (v) Critical path.
14. What is critical path method (CPM) ? The programme evalution and review technique (PERT) ? What does each involve ? How are they similar ? Different ? What particular advantages does *PERT* have over *CPM* ? Why is this an advantage for the operations manager ?
15. Give your comments on *PERT* and *CPM.*
16. Briefly explain the following with examples in relation of network analysis :
(i) Crashing, (ii) Resource Allocations.
17. Explain with illustrations, possible type of resource levelling.
18. State the circumstances where CPM is a better technique of project management than PERT. **[Delhi (M.Com.) 90, 89]**
19. How does the PERT technique help a business manager in decision making. **[Delhi (M.Com.) 90]**
20. Critically comment on the assumptions based on which *PERT/CPM* analysis is done for the projects. **[C.A. (May) 91]**

EXAMINATION REVIEW PROBLEMS

1. For the following data, draw network. Find the critical path, slack time after calculating the earliest expected tine and the latest allowable time :

Activity	Duration	Activity	Duration
(1-2)	5	(5–9)	3
(1–3)	8	(6–10)	5
(2–4)	6	(7–10)	4
(2–5)	4	(8–11)	9
(2–6)	4	(9–12)	2
(3–7)	5	(10–12)	4
(3–8)	3	(11–13)	1
(4–9)	1	(12–13)	7

[**Ans.** $E_1 = 0$, $L_1 = 0$; $E_2 = 5$, $L = 9$; $E_3 = 8$, $L_3 = 8$; $E_4 = 11$, $L_4 = 18$; $E_5 = 9$, $L_5 = 16$; $E_6 = 9$; $L_6 = 12$; $E_7 = 13$, $L_7 = 13$; $E_8 = 11$, $L_8 = 18$; $E_9 = 12$, $L_9 = 19$; $E_{10} = 17$, $L_{10} = 17$; $E_{11} = 20$, $L_{11} = 27$; $E_{12} = 21$, $L_{12} = 21$; $E_{13} = 28$, $L_{13} = 28$. Critical path : 1 → 3 → 7 → 10 → 12 → 13, Event slack : 0, 3, 0, 7, 7, 3, 0, 7, 7, 0, 7, 0, 0.]

2. A small project is composed of 7 activities whose time estimates are listed in the table below. Activities are identified by their beginning (i) and ending (j) node numbers.

Activity (i–j)	Estimated Duration (Weeks)		
	Optimistic	Most likely	Pessimistic
(1–2)	1	1	7
(1–3)	1	4	7
(1–4)	2	2	8
(2–5)	1	1	1
(3–5)	2	5	14
(4–6)	2	5	8
(5–6)	1	6	15

(*i*) Draw the project network and identify all paths through it.

(*ii*) Find expected duration and variance for each activity.

(*iii*) Calculate early and late occurrence time for each node. What is expected project length ?

(*iv*) Calculate the variance and standard deviation of the project length. What is the probability that the project will be completed,

(*i*) at least 4 weeks earlier than expected time; (*ii*) no more than 4 weeks later than expected time ?

(*v*) If the project due date is 18 weeks, what is the probability of not meeting the due date.

Given	:						
		Z :	0.50	0.67	1.00	1.33	2.00
		P :	0.3085	0.2514	0.1587	0.0918	0.0228

[Gujarat (M.B.A.) 90]

3. A project schedule has the following characteristics.

Activity i–j	Most optimistic time	Most likely time	Most pessimistic time
(1–2)	1	2	3
(2–3)	1	2	3
(2–4)	1	3	5
(3–5)	3	4	5
(4–5)	2	5	4
(4–6)	3	5	7
(5–7)	4	5	6
(6–7)	6	7	8
(7–8)	2	4	6
(7–9)	4	6	8
(8–10)	1	2	3
(9–10)	3	5	7

Construct a *PERT* network and find out :

(*i*) The earliest possible time (T_e) to complete the different stages of the project.

(*ii*) The latest allowable time (T_L) for them, (*iii*) The slack values, (*iv*) The critical paths,

(*v*) The probability factor for completing the project in 30 weeks.

4. (a) What do you understand by : (*i*) Criticasl and semi-critical path, (*ii*) σ for a network, (*iii*) Slack time,

(*iv*) Probability of finishing an activity within scheduled time.

(b) Number the given network (*Fig.* 25.76) : determine the critical path and the probability of finishing the project within the scheduled time T_s – 35,

Assume :

probability (= normal deviate = 50 + 35 × (± normal deviate)

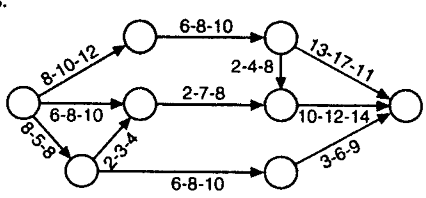


Fig. 25.77

5. Given the following "system flow plan".

Also, given that the original schedule of the completion of the project is 8 hours. The first coordinate on the activities stands for the expected value of the activity times and the second coordinate stands for the variance of the activity time in hours. Determine the following : (*i*) Critical path for the project. (*ii*) Probability of completing the project in time.

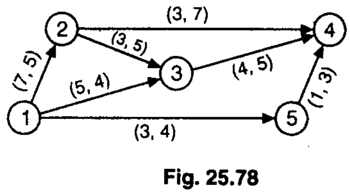


Fig. 25.78

6. A small project is composed of seven activities where time estimates are listed in the table below. Activities are identified by their beginning (i) and ending (j) node numbers.

Activity i–j	Optimistic	Most likely	Pessimistic
(1–2)	0.7	1.0	1.3
(2–3)	3.8	5.6	9.8
(2–4)	5.2	7.6	12.4
(3–4)	2.1	2.7	6.1
(4–5)	0.7	3.4	3.7
(5–6)	0.7	1.0	1.3

(*i*) Find the expected duration and standard deviation for each activity.

(*ii*) What is the probability that the project will be completed two weeks earlier than expected ?

7. Normal times, crash times and costs are given below for the following project.

Activity (*i–j*)	Time (in days)		Cost (in Rupees)	
	Normal	Crasth	Normal	Crash
(1–2)	8	3	7,000	10,000
(1–3)	4	2	6,000	8,000
(2–3)	0	0	0	0
(2–5)	6	1	9,000	11,500
(3–4)	7	5	2,500	3,000
(4–6)	12	8	10,000	16,000
(5–6)	15	10	12,000	16,000
(5–7)	7	6	12,000	14,000
(6–8)	5	5	10,000	10,000
(7–8)	14	7	6,000	7,400
(7–9)	8	5	6,000	12,000
(8–9)	6	4	6,000	7,800

Indirect cost per day is Rs. 1,000/-.

Draw the network diagram and determine, by trading-off between time and cost parameters, the optimum project completion time and the minimum total cost of the project.

Draw the network diagram for a projecrt consisting of 12 tasks (*A, B, ..., L*) in which the following precedence relationship must hold (*X* < *Y* means *X* must be completing before *Y* can start)

A < *C*; *A* < *B*; *B* < *D*; *B* < *G*; *B* < *K*; *C* < *D*; *A* < *G*; *D* < *E*; *E* < *F*; *F* < *H*; *F* < *I*; *F*< *L*; *G* < *I*; *G* < *L*; *H*< *J*; *I* < *J*; and *K* < *L*.

Given the following task times for the above project, locate the critical path :

Task	A	B	C	D	E	F	G	H	I	J	K	L
Time	30	7	10	14	10	7	21	7	12	15	30	15

Find also the free and total floats for the non-critical activities.

9. Briefly explain *PERT/time* and *PERT/Cost*. Given below is a list of activities and sequencing requirements as indicated, whcih comprise necessary activities for the completion of a thesis.

Activities	Description	Prerequisite Activity	Expected time (weeks)
a	Literature search	None	5
b	Topic formulation	None	5
c	Committee selection	b	2
d	Formal proposal	c	2
e	Company selection and contact	a, d	2
f	Progress report	d	1
g	Formal research	a, d	6
h	Data collection	e	5
i	Data analysis	g, h	6
j	Conclusions	i	2
k	Rough draft	g	4
l	Final copy	j, k	3
m	Viva examination	*l*	1

(a) Draw a network diagram of this project, (b) List the activities which are on the critical path,

(c) What is the minimum project completion time.

10. The following figure gives a CPM network for a project in arrow notation in which durations are given in number of weeks. Compute for each job :
(*i*) Earliest start time,
(*ii*) Earliest finish time,
(*iii*) Latest start time,
(*iv*) Latest finish time,
(*v*) Total float
(*vi*) Free Float,
(*vii*) Independent float.

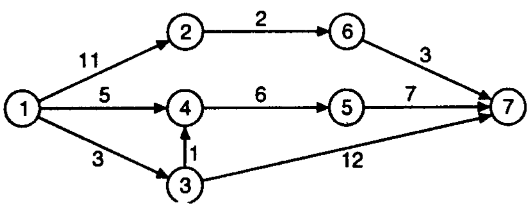


Fig. 25.79

11. Describe the following by network :

$A \xrightarrow{2} B,\ A \xrightarrow{3} C,\ B \xrightarrow{4} C,\ B \xrightarrow{3} D,\ C \xrightarrow{3} F,\ D \xrightarrow{12} E,\ E \xrightarrow{3} G,\ E \xrightarrow{5} H,\ F \xrightarrow{8} H,\ G \xrightarrow{0} H.$

There is a constraint that activity $F \to H$ cannot start till the activity $D \to E$ is completed. Determine the critical path and tabulate the earliest start time, earliest finish time, latest start time, latest finish time, total, float free float.

12. A research and development department is developing a new power supply for a console television set. It has broken the job down into following form :

Job	Description	Immediate Predecessor	Time (days)
a	Determine output voltages	–	5
b	Determine whether to use solid state rectifier	a	7
c	Choose rectifiers	b	2
d	Choose filters	b	3
e	Choose transformers	c	1
f	Choose chassis	d	2
g	Choose rectifier mounting	c	1
h	Layout chassis	e, f	3
i	Build and test	g, h	10

(*i*) Draw a critical-path scheduling-arrow diagram, identifying jobs by letters and associating times with each, indicate the critical path.

(*ii*) What is the minimum time for completion of the project.

13. A company manufacturing plant and equipment for chemical processing is in the process of quoting a tender called by a *Public Sector Undertaking*. Delivery date once promised is crucial and penalty clause is applicable. The winning of tender also depends on how soon the company is able to deliver the goods. *Project Manager* has listed down the activities in the project as under :

Sl.No.	Activity	Immediate preceding activity	Activity time (week)
1	A	–	3
2	B	–	4
3	C	A	5
4	D	A	6
5	E	C	7
6	F	D	8
7	G	B	9
8	H	E, F, G	3

(*i*) Find out the delivery week from the date of acceptance of quotation,

(*ii*) Find out the total float, free float, and independent float for each of the activiry.

14. A reactor and storage tank are interconnected by a 3" insulated process line that needs periodic replacement. There are valves along the lines and at the the terminals and these need replacing as well. No pipe and valves are in stock. Accurate, as built, drawings exist and are available. The line is overhead and requires scaffolding. Pipe sections can be ship fabricated at the plant. Adequate craft labour is available. You are the maintenance and construction superintendent responsible for this project. The works engineer has requested your plan and schedule for a review with the operating supervision. The plant methods and standards section has furnished the following data. The precedents for each activity have been determined from a familiarity with similar projects.

Symbol	Activity Description	Time (Hrs.)	Precedents
A	Develop required material list	8	–
B	Procure pipe	200	A
C	Erect scaffold	12	–
D	Remove scaffold	4	I, M
E	Deactivate line	8	–
F	Prefabricate sections	40	B

G	Place new pipes	32	F, L
I	Fit up pipe and valves	8	G, K
J	Procure valves	225	A
K	Place valves	8	J, L
L	Remove old pipe and valves	35	C, E
M	Insulate	24	G, K
N	Pressure test	6	I
O	Clean-up and start-up	4	D, N

(*i*) Sketch the arrow diagram of this project plan.

(*ii*) Make the forward pass and backward calculations on this network, and indicate the critical path and its length.

(*iii*) Calculate total float and free float (both early and late) for each of the non-critical activities.

15. Draw the network of the following activities and tabulate earliest and latest starting and finishing time of each activity and the total and free floats of them :

Activity	Description	Duration (days)
(1–2)	Excavation	4
(1–3)	Order and delivery of steel	17
(2–3)	Framework of steel	4
(2–4)	Foundation	5
(3–4)	Dummy	0
(3–5)	Concrete work	8
(4–6)	Placement of frames	2
(5–6)	Dummy	0
(5–9)	Back filling	3
(6–7)	Concrtete-stage-2	8
(7–8)	Dummy	0
(7–9)	Dummy	0
(8–10)	Steel work stage-2	10
(9–10)	Block filling stage-2	5

[Osmania (MBA) 90]

16. The following table gives for each activity of a project, its duration and corresponding resource requirements as well as total availability of each type of resources :

Activity	Duration (days)	Resources required	
		Machines	Men
(1–2)	7	2	20
(1–3)	7	2	20
(2–3)	8	3	30
(2–4)	6	4	30
(3–6)	9	2	20
(4–5)	3	2	20
(5–6)	5	2	20
Minimum available resources		4	40

(*i*) Draw the network, compute Earliest occurrence time, and Latest occurrence time for each event the total float for each activity and identify the critical path assuming that there are no resource constraints.

(*ii*) Under the given resource constraints find out the minimum duration to complete the project and compare the utilization of the resources for that duration. **[I.C.W.A. Dec. 90]**

17. The required data for a small project consisting of different activities are given below :

Activity	Dependence	Normal duration (days)	Normal cost (Rs.)	Crash duration (days)	Crash cost (Rs.)
A	–	6	300	5	400
B	–	8	400	6	600
C	A	7	400	5	600
D	B	12	1000	4	1400
E	C	8	800	8	800
F	B	7	400	6	500
G	D, E	5	1000	3	1400
H	F	8	500	5	700

(*i*) Draw the network and find out the normal project length and minimum project length.

(*ii*) If the project is to be completed in 21 days with minimum crash cost, which activities should be crashed by how many days ? **[I.C.W.A. June 90]**

18. The following table gives the activities in a construction project and other relevant informations :

Activity	:	(1–2)	(1–3)	(2–3)	(2–4)	(3–4)	(4–5)
Duration	:	20	25	10	12	6	10

(*i*) Find free, total and independent floats for each activity, (*ii*) Determine th ecritical path.

[VTU (BE Mech.) 2002; Meerut (I.P.M.) 90]

19. (a) Compare the two techniques *PERT* and *CPM*. **(4 marks)**

[I.C.W.A. June 91, 85; Madras (M.Com.) 90]

(b) What are the assumptions that are underlying in the *CPM* analysis. **(4 marks)**

(c) Give one example of one area where *CPM* technique is applied. **(1 mark)**

(d) The following table gives the various activities, their duration, direct costs. The indirect cost is Rs. 2,000 per week. Find the minimum cost schedule using *CPM* technique. **(10 marks)**

Activity (*i–j*)	Time (in week)		Cost (in Rs.)		Cost (in Rs.) to expedite per week
	Normal	Crash	Normal	Crash	
(1–2)	8	4	3,000	6,000	750
(1–3)	5	3	4,000	8,000	2,000
(2–4)	9	6	4,000	5,500	500
(3–5)	7	5	2,000	3,200	600
(2–5)	5	1	8,000	12,000	1,000
(4–6)	3	2½	10,000	11.200	2,400
(5–6)	6	2	4,000	6,800	700
(6–7)	10	7	6,000	8,700	900
(5–7)	9	5	4,200	9,000	1,200
Total			45,200	70,400	

(e) Does your answer very, if the objective is to minimize the project duration ? In that case what is the cost and duration. **(1 mark)**

[I.C.W.A. June 91]

20. The activities of a project are tabulated below with immediate predecessors and normal & crash time cost.

Activity	Immeidate predecessor	Normal		Crash	
		Cost (Rs.)	Time (days)	Cost (Rs.)	Time (days)
A	—	200	3	400	2
B	—	250	8	700	5
C	—	320	5	380	4
D	A	410	0	800	4
E	C	600	2	670	1
F	B, E	400	6	950	1
H	B, E	550	12	1,000	6
G	D	300	11	400	9

(i) Draw the network corresponding to Normal time.
(ii) Determine the critical path and the normal duration and cost of the project.
(iii) Suitably crash the activities so that the normal duration may be reduced by 3 days at minimum cost. Also find the project cost for this shortened duration if the indirect cost per day is Rs. 25. **[Gujarat (M.B.A.) 95]**

[**Hint.** Critcal path : $A \to D \to H$ (or $1 \to 2 \to 5 \to 7$)
Normal duration = 23 days
Normal cost = sum of normal cost of all the activities + Indirect cost
= Rs. 3030 + Rs. 25 × 23 = Rs. 3605]

21. A marketing manager wants to launch a new product. The table below shows jobs, their normal time and crash time and cost for the project.

Job	Normal		Crash	
	Time (days)	Cost (Rs.)	Time (days)	Cost (Rs.)
(1—2)		1,400	4	1,900
(1—3)	8	2,000	5	2,800
(2—3)	4	1,000	2	1,500
(2—4)	3	800	2	1,400
(3—4)	Dummy	—	—	—
(3—5)	6	900	3	1,600
(4—6)	10	2,500	6	3,500
(5—6)	3	500	2	800

Indirect cost for the project is Rs. 300 per day.
(i) Draw the network of the project.
(ii) What is the normal duration and cost of the project ?
(iii) If all activities are crashed, what will be the project duration and corresponding cost ?
(iv) Find the optimum duration and minimum project cost. **[Punjabi (M.B.A.) Dec. 96]**
[**Hint.** (ii) 20 days and Rs. 9,200.
(iii) Following table gives the information regarding optimum schedule in terms of crashing and direct cost.

Normal project length (days)	Crashing time and cost (days/Rs.)	Indirect cost	Total cost
20	—	20 × 300	6,000
19	1 × 200 = 200	19 × 300	5,900
18	1 × 250 = 250	18 × 300	5,650
17	1 × 250 = 250	17 × 300	5,350
16	1 × 200 + 1 × 600 + 1 × 233 = 1033	16 × 300	5,833

22. The relevant data for a project are given below :

Activity	Preceding activity	Time (in weeks)		Cost (Rs.)	
		Normal	Crash	Normal	Crash
A	None	12	10	5,000	7,400
B	A	9	8	4,300	5,900
C	A	6	4	3,700	4,900
D	C	3	3	2,500	2,500
E	B	7	6	3,900	4,800
F	D	8	5	4,600	6,100
G	D, E	4	4	2,800	2,800
H	F	6	3	4,100	6,200
I	G	7	5	4,800	7,800

This project must be completed in 34 weeks. Suggest the least-cost schedule for completing the project.
[Delhi (M. Com.) 95]
[**Ans.** Critical path : $A \rightarrow B \rightarrow E \rightarrow G \rightarrow I$ and the project duration = 39 weeks, and cost of completion is Rs. 35,700.]

23. In planning a project to introduce a new product in the market, a company lists the various activities, their normal times & costs and their crash times & costs as shown in the table below :

Activity	Immediate predecessor	Normal		Crash	
		Time (T_n)	Cost (C_n)	Time (T_c)	Cost (C_c)
A	—	5	10,000	4	12,000
B	—	2	6,000	2	6,000
C	A	4	8,000	3	10,000
D	A	4	10,000	3	15,000
E	A	3	11,000	1	16,000
F	C	1	7,000	1	7,000
G	D	4	8,000	2	12,000
H	B, E	5	9,000	3	12,000
I	H	2	8,500	2	8,000
J	F, G, I	3	7,500	2	10,000

[Delhi (M.B.A.) Dec. 94]

24. The time-cost estimates for the various activities of a project are given below :

Activity	Preceding activity	Time (in weeks)		Cost (in Rs.)	
		Normal	Crash	Normal	Crash
A	—	8	6	8,000	10,000
B	—	7	5	6,000	8,400
C	A	5	4	7,000	8,500
D	B	4	3	3,000	3,800
E	A	3	2	2,000	2,600
F	D, E	5	3	5,000	6,600
G	C	4	3	6,000	7,000

The project manager wishes to complete the project in the minimum possible time. However, he is not authorised to spend more than Rs. 5,000 on crashing.

Suggest the least-cost schedule for achieving the objective of the project manager. Assume that there is no indirect or utility cost. **[Delhi (M. Com.) 96]**

[Hint.

Project duration	Direct cost	Indirect cost	Total cost
18	85,000	72,000	1,57,000
16	88,000	64,000	1,52,000
15	90,000	60,000	1,50,000
14	92,500	56,000	1,48,500
13	97,000	52,000	1,42,000
12	1,01,000	48,000	1,49,500

25. Suggest optimum crashing schedule for the following project :

Activity	Preceding activity	Time (in weeks)		Cost (in Rs.)	
		Normal	Crash	Normal	Crash
A	—	7	4	2,100	3,000
B	—	5	3	1,400	1,800
C	—	8	5	2,700	3,900
D	A	2	1	900	1,0[illegible]
E	B	3	1·5	1,200	1,500
F	C	3	1·5	1,000	1,300
G	D, E, F	4	2	1,200	1,700

[Delhi (M. Com.) 98]

26. The table below provides cost and give estimates of seven activities of a project :

Activity (*i* – *j*)	Time estimates (weeks)		Direct cost estimates (Rs. in thousands)	
	Normal	Crash	Normal	Crash
(1—2)	2	1	10	15
(1—3)	8	5	15	21
(2—4)	4	3	20	24
(3—4)	1	1	7	7
(3—5)	2	1	8	15
(4—6)	5	3	10	16
(5—6)	6	2	12	36

(i) Draw the project network corresponding to normal time.
(ii) Determine the critical pathg and the normal duration and normal cost of the project.
(iii) Crash the activities so that the project completion time reduces to 9 weeks, with minimum additional cost.

[C.A. (May) 92]

27. A small project is having seven activiites. The relevant data about these activities is given below :

Activity	Dependence	Normal duration (days)	Crash duration (days)	Normal cost (Rs.)	Crash cost (Rs.)
A	—	7	5	500	900
B	A	4	2	400	600
C	A	5	5	500	500
D	A	6	4	800	1,000
E	B, C	7	4	700	1,000
F	C, D	5	2	800	1,400
G	E, F	6	4	800	1,600

(i) Find out the normal duration and the minimum duration.

(ii) What is the percentage increase in cost to complete the project in 21 days ? **[C.A. (May) 97]**

28. A company has recently won a contract for the installation of a die casting machine and its associated building construction work at a local factory of a large national firm of electronic engineers. The following table gives the various activities involved in this kjob, their normal time and cost estimates and theircrash time and cost estimates.

Activity	Description	Predecessor	Normal		Crash	
			Time (days)	Cost (Rs.)	Time (days)	Cost (Rs.)
A	Prpeare foundations and underground services and erect building frame structure	—	30	90,000	25	1,05,000
B	Fabric part and assemble steel frames to support the machine	—	25	1,80,000	20	1,90,000
C	Collect die casting machine and its associated gear from the manufacturers	—	10	50,000	8	54,000
D	Assemble and check control gear.	C	10	7,500	7	9,000
E	Fit control gear on steel frames and sinstal.	B, D	10	4,200	10	4,200
F	Fit aluminium sheet wall claddings.	A, E	20	20,000	16	30,000
G	Erect assembled plant on to prepared foundation and frame work and connect services.	A, E	35	28,000	30	35,500
H	Erect mechanical handling plant.	B, D	20	12,000	18	15,000
I	Fit ventilation and fire protection system	F	20	14,000	15	24,000

If the variable overhead costs are Rs. 5,000 per day, determine the optimum project duration. **[Delhi (M.B.A.) 95]**

29. The following table gives the list of various activities involved in the production of a wireless communication equipment, their immeidater predecessor, their normal time andf cost estimates and their crash time and cost estimates :

Activity	Activity description	Time (months)		Cost (Rs.)		Immediate
		Normal	Crash	Normal	Crash	predecessor(s)
a	System calcualtions	3	1	45,000	63,000	—
b	Release of drawings	2	1	15,000	23,000	—
c	Procurement of PSU	12	10	85,000	1,05,000	*a, b*
d	Procurement of raw material	8	6	4,50,000	5,00,000	*a, b*
e	Production documentation	4	3	40,000	49,500	*a, b*
f	Time study/shop order	3	1	25,000	41,000	*e*
g	PCB Manufacture	6	5	50,000	59,000	*d, f*
h	Mechanical parts manufacture	7	6	2,00,000	2,20,000	*d, f*
i	Electronic Assembly	2	2	43,000	43,000	*c, g, h*
j	Testing	4	2	50,500	75,000	*i*

[Delhi (MBA) Dec. 94]

30. An electronics firm has signed a contract to instal an instrument landing device at the local airport. The complete installation can be broken down into fourteen separate activities. Each activity (labeled *A* through *N*), its predecessor activities, normal times and cost, and crash time and cost are given below. The contract specifies that the installation will be completed within 18 days. There is a penalty of Rs. 10,000 per day beyond the specified completion time.

Activity	Predecessor activities	Normal time (days)	Normal cost (Rs.)	Crash time (days)	Crash cost (Rs.)
A	—	3	32,000	2	36,000
B	—	5	55,000	4	50,000
C	—	6	57,500	4	70,000
D	A	7	75,000	5	85,000
E	A	4	42,000	3	49,000
F	B, D	2	18,000	2	18,000
G	C	4	42,500	4	48,500
H	A	8	85,000	5	1,06,000
I	C	5	57,500	4	53,500
J	C	7	67,500	5	73,500
K	E, F, G	4	40,000	3	44,000
L	H, I	6	65,000	4	75,000
M	L	3	28,000	2	33,500
N	J, K	5	52,500	4	57,500

(i) What is the normal time to complete the installation ?
(ii) What is the shortest possible time for completion of the installation ?
(iii) What is the most economical period of time in which to schedule the installation ?
(iv) What is the minimum total cost (installation plus penalty) ? **[Bombay (M.M.S.) 96]**

31. A maintenance project consists of the jobs given in the following table. With each job is listed its normal time & crash and normal cost & crash cost.

Activity	Immeidate predecessor(s)	Normal		Crash	
		Time (days)	Cost (Rs.)	Time (days)	Cost (Rs.)
A	—	15	1,500	5	1,500
B	—	15	7,200	10	8,000
C	—	30	8,400	18	9,000
D	—	20	2,100	14	2,700
E	A	12	1,400	8	1,560
F	A	6	800	4	1,200
G	E	24	6,800	20	7,800
H	F	8	1,000	5	1,240
I	F	4	600	3	900
J	B, F	10	3,000	7	3,450
K	G, H	11	2,500	8	3,580
L	E, J, K	9	1,800	6	2,700
M	C, J	14	2,600	10	3,320
N	D, J	21	8,400	15	10,800
O	L, M	10	1,900	6	2,140
P	M, N	12	1,300	10	1,400
Q	K, N, O	7	700	5	840
R	P, E	3	500	3	500

Indirect cost per day in Rs. 200.

(a) Check and remove redundant immediate predecessors, if any, (b) Draw an arrow diagram, (c) Determine optimum time versus cost schedule. **[Delhi (M.B.A.) March, 99]**

32. From the data given below, construct the network and number the nodes using Fulkerson's rule. Calculate the expected task times and their variance. Carrying out the forward pass computation and backward pass computation, find T_E and T_L for all nodes.

Task	A	B	C	D	E	F	G	H	I	J	K
Least time (days)	4	5	8	2	4	6	8	5	3	5	6
Maximum Time (days)	8	7	12	7	10	15	16	9	7	11	13
Most likely time (days)	5	7	11	3	7	9	12	6	5	8	9

Precedence relationship : A, C, D can start simultaneously, $E > B, C$; $F, G > D$; $H, I > E, F$; $J > I, G$; $K > H$; $B > A$.

Also, determine (i) critical path.

(ii) Probability of completing the project in 40 days.

(iii) Duration of the project if the probability of comptetion = 0·816. **[VTU (BE Mech.) 2002]**

33. What are the requirements for the application of PERT ? Give an algorithm for PERT and state the limitations of this technique. **[Meerut (OR) 2003]**

34. A multinational FMCG company wishes to launch a new Fruit Yogurt in the coming season. A brief description of the activities associated with this project, their expected durations (in weeks) and their immediate predecessor(s) are given in the following table :

Activity	Description	Predecessor	Expected Time (Week)		
			Optimistic	Likely	Pessimistic
A	Management approval	—	2	2.5	4
B	Product concept test	A	3	4.7	5
C	Technical feasibility	A	2	2	3
D	Recipe finalization	C, B	1	1	2
E	Shelf life trials	D	8	12	15
F	Brand positioning study	B	4	5	7
G	Packaging key lines	F	1	1	2.5
H	Agency advertisement development	F, E	4	8	10
I	Agency layouts artworks	G	2	3	5
J	Advertisement test research	H	3	4	5
K	Cost finalization	D	1	1	2
L	Pricing decision	J, F	1	1.5	3
M	Marketing max finalization	L	2	2.5	3
N	POS development	M	2	4	5
O	Launch plans	M	1	1	1.5
P	Branch communication	O	0.5	0.5	1
Q	Supplier's delivery of packaging	H	4	5	8
R	Production trail	D, Q	1.5	2	3
S	Management final approval	R	0.5	1	1.5
T	Final production	S	1	2.5	3
U	Stock movement	T	0.5	1.5	2
V	Position movement	T	0.5	1	1.5
W	Launchs	U, V	1	2	4

Management of the company desires to know the realistic completion time for this project and detailed analysis of float times (if any). **[Delhi (MBA) 2000]**

OBJECTIVE QUESTIONS

1. The objective of network analysis is to
   (a) minimize total project duration. (b) minimize total project cost.
   (c) minimize production delays, interruption and conflicts. (d) all of the above.
2. Network models have advantage in terms of project
   (a) planning. (b) scheduling. (c) controlling. (d) all of the above.

3. The slack for an activity is equal to
   (a) LF – LS. (b) EF – ES. (c) LS – ES. (d) none of the above.
4. The another term commonly used for activity slack time is
   (a) total flot. (b) free float. (c) independent float. (d) all of the above.
5. Generally PERT technique deals with the project of
   (a) repetitive nature. (b) non-repetitive nature. (c) deterministic nature. (d) none of the above.
6. In PERT the span of time between the optimistic and pessimistic time estimates of an activity is
   (a) 3 σ. (b) 6 σ. (c) 12 σ. (d) none of the above.
7. If an activity has zero slack, it implies that
   (a) it lies on the critical path. (b) it is a dummy activity.
   (c) the project progressing well. (d) none of the above.
8. A dummy activity is used in the network diagram when
   (a) two parallel activities have the same tail and head events.
   (b) The chain of activities may have a common event yet be independent by themselves.
   (c) both (a) and (b).
   (d) none of the above.
9. While drawing the network diagram, for each activity project, we should look
   (a) what activities precede this activity?
   (b) what activities follow this activity?
   (c) what activities can take place concurrently with this activity?
   (d) all of the above.
10. In PERT network each activity time assume a Beta distribution because
   (a) it is a unimodal distribution that provides information regarding the uncertainty of time estimates of activities.
   (b) it has got finite non-negative error.
   (c) it need not be symmetrical about model value.
   (d) all of the above.
11. The critical path satisfy the condition that
   (a) $E_i = L_i$ and $E_j = L_j$. (b) $L_j - E_i = L_i - L_j$.
   (c) $L_j - E_i = L_i - E_j = c$ (constant). (d) all of the above.
12. Float or slack analysis is useful for
   (a) projects behind the schedule only. (b) projects ahead of the schedule only.
   (c) both a and b. (d) none of the above.
13. The activity which can be delayed without affecting the execution of the immediate succeeding activity is determined by
   (a) total float. (b) free float. (c) independent float. (d) none of the above.
14. In time-cost-trade-off function analysis
   (a) cost decreases linearly as time increases. (b) cost at normal time is zero.
   (c) cost increases linearly as time increades. (d) none of the above.

Answers

1. (a) **2.** (d) **3.** (c) **4.** (d) **5.** (d) **6.** (b) **7.** (a) **8.** (c) **9.** (a) **10.** (a) **11.** (a) **12.** (a). **13.** (b) **14.** (a).

***

26

INFORMATION THEORY

26.1. INTRODUCTION

In everyday life, we observe that there are numerous means for the transmisson of information. For example, the information is usually transmitted by means of the human voice (as in telephone, radio and television), by means of letters, newspapers, books etc. We often come across sentences like :

(*i*) We have received a *lot of information* about the postponment of examinations.

(*ii*) We have a *bit of information* that he will be appointed as a Professor.

But, few people have suspected that it is really possible to measure information quantitatively. Nevertheless, an amount of information has a useful numerical value just like an amount of sugar or the amount of a bank balance.

Before introducing numbers of formulae, let us consider a few examples.

Example 1. Suppose we are listening to the local weather forecast on the radio on first of July and we hear the weatherman, say, "Monsoon will come during this week". Since it has probably not rained during Ist week of July for a decade or more, we would consider that such a statement by the weatherman give us a great deal of information. If, however, we know the weatherman to be very reliable, and we heard him say, "Monsoon will not come during this week", then we would consider that such a statement by the weatherman give us a very little information. *Thus, according to our usual way of looking at information, if something is very likely to occur, the statement that it will occur does not give much information. On the other hand, if something is unlikely to occur, the statement that it will occur gives a good deal of information.*

Example 2. Suppose a man goes to a new community to rent a house and asks an unreliable agent, "Is this house cool in the summer season ?" If the agent answers, *"Yes"*, the man has received very little information because more than likely the agent would have answered, *"Yes"* regardless of the facts. If on the other hand, the man has a friend who lives in the neighbouring house, he can get more information by asking his friend the same question because the answer will be more reliable.

In a general way, it would appear that the amount of information in the message should be measured by the extent of the change in probability produced by the message.

The subject *'Information Theory'* is about four decades old. This is a new branch of probability theory with a large number of applications to communication systems. The scientists, while studying the statistical structure of electrical communication equipment, originated this subject. Mathematical theory of communications was principally initiated by ***Claude Shannon*** in 1984.

We now proceed to have a brief discussion about this new subject.

26.2. COMMUNICATION PROCESSES

Definition. *The communication process may be defined as the procedure by which one mind affects another is called a* ***communication process.*** *This may be any means by which the information is carrried from a transmitter to receiver.*

The flow of some commodity in some network is common to all communication processes. The nature of the commodity can be as varied as electricity, words, pictures, music, etc. So there will be at least three essential parts (*Fig* 26.1) of a simplest communication system such as

1. Transmitter or source

2. Receiver or sink
3. Channel or transmission network which carries the message form the transmitter to receiver.

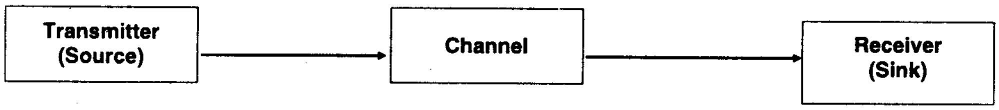


Fig. 26.1.

26.3. MODEL FOR A COMMUNICATION SYSTEM

It is traditional to represent a communication system by a diagram (*Fig.* 26.2).

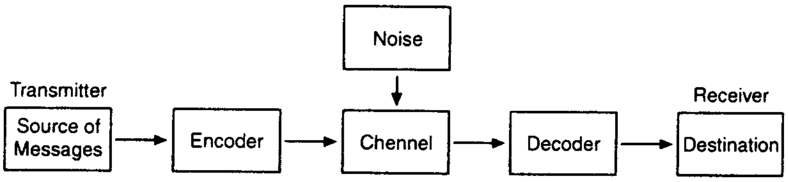


Fig. 26.2 General structure of a communication system

Each part of the communication system is explained below :

1. Transmitter. The transmitter (or the source of messages) is the person or machine which produces the information to be communicated.

2. Encoder. Generally the transformer is used to improve the efficiency of the electrical system, but in terms of information theory, the device, which is used to improve the efficiency of the medium through which the message is transmitted, is called *Encoder.* Encoder acts as step-up transformer.

3. Channel. The *channel* is the medium over which the coded message is transmitted.

4. Decoder. At the receiver's terminal, a decoder is employed to transform encoded message into the original form which is acceptable to the receiver.

Decoding apparatus acts as step-down transformer.

5. Receiver. Receiver is the destination to which the message is transmitted.

6. Noise. In general, the message cannot be transmitted with complete reliability because of effects of noise. The *'noise'* is a general term for anything which tends to produce errors in transmission, such as noise in radio and television or passage of a vehicle in the opposite direction in one way traffic.

In fact, information theory is an attempt to formulate a mathematical model for each of six parts of a communication system.

26.4. FUNDAMENTAL THEOREM OF INFORMATION THEORY

The information theory is essentially a study of so-called "*fundamental theorem of information theory*" which states that—

"*It is possible to transmit information through a noisy channel at any rate less than the channel capacity with an arbitrarily small probability of error*".

26.5. STATISTICAL NATURE OF COMMUNICATION SYSTEMS

The communication system considered here is statistical in nature. A source is a device that selects and transmits sequences of symbols from a given alphabet, each selection is made at random, although this section may be based on some statistical rule. The channel transmits incoming symbols to the receiver. The performance of the channel is also based on laws of chance. If the source transmits a symbol say E, with a probability denoted by $P(E \mid E)$, then the probability of transmitting E and receiving E is

$$P(E) : P(E \mid E). \qquad ...(26.1)$$

26.5–1. Memoryless Channel, Binary Symmetric Channel

Definition. (*Memoryless Channel*). *A memoryless channel* is described by an input alphabet $A = \{x_1, x_2, \dots, x_m\}$, and output alphabet $B = \{y_1, y_2, \dots, y_n\}$, and a set of conditional probabilities $P(y_j \mid x_i)$ for all i and j, where $P(y_j \mid x_i)$ is the probability that the output symbol y_j will be received in, the input symbol x_i is sent.

Note. For simplicity, the word 'channel' will be used in place of 'memoryless channel' because, only the memoryless channels will be discussed in this chapter.

Definition. (*Binary Symmetric Channel*). *A binary symmetric channel* has just two input symbols ($x_1 = 0, x_2 = 1$) and two output symbols ($y_1 = 0, y_2 = 1$), and it is symmetric in the sense that

$$P(y_1 \mid x_1) = P(y_2 \mid x_2) = \bar{p}, \quad P(y_1 \mid x_2) = P(y_2 \mid x_1) = p$$

where $\bar{p} = 1 - p$, p being the probability of error in transmission.

26.5–2. The Channel Matrix

A simple way of representing a channel is to arrange the output conditional probabilities as shown in the following *channel matrix* :

$$\text{Input} \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_m \end{array} \overset{\text{Output}}{\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array} \\ \begin{bmatrix} p_{1|1} & p_{2|1} & \cdots & p_{n|1} \\ p_{1|2} & p_{2|2} & \cdots & p_{n|2} \\ \vdots & \vdots & & \vdots \\ p_{1|m} & p_{2|m} & \cdots & p_{n|m} \end{bmatrix} \end{array}}$$

where $p_{j|i} = P(y_j \mid x_i)$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$.

Each row of the channel matrix corresponds to an input of the channel, and each column corresponds to a channel output.

Note that in every channel matrix, the sum of terms in every row must be equal to one, *i.e.*, $\sum_{j=1}^{n} p_{j|i} = 1, \; i = 2, \dots, m.$

For example, the channel matrix of the *Binary Symmetric Channel is* given by $\begin{pmatrix} \bar{p} & p \\ p & \bar{p} \end{pmatrix}$

26.5–3. Probability Relation in a Channel

Let us consider a channel with m input symbols $x_1, x_2, \dots, x_m$ and n output symbols $y_1, y_2, \dots, y_n$; and the channel matrix

$$\begin{bmatrix} p_{1|1} & p_{2|1} & \cdots & p_{n|1} \\ p_{1|2} & p_{2|2} & \cdots & p_{n|2} \\ \vdots & \vdots & & \vdots \\ p_{1|m} & p_{2|m} & \cdots & p_{n|m} \end{bmatrix}$$

If $p_{i0} = P(x_i)$, $i = 1, 2, \dots, m$ denote the probability that symbol x_i will be selected for transmission through the channel and $p_{0j} = P(y_j)$, $j = 1, 2, \dots, n$ denote the probability that output symbol y_j will be received as channel output. Then the relation between the probabilities of various input symbols and output symbols may be obtained. The following relations may be easily obtained

$$\sum_{i=1}^{m} p_{i0}\, p_{j|i} = p_{0j} \text{ for } j = 1, 2, \dots, n. \qquad \dots(i)$$

If we are given the input probabilities p_{i0} and the channel probabilities, then the output probabilities p_{0j} can be easily obtained from above relation.

Furthermore, the following probability relations also hold :

$$P(x_i, y_j) = p_{j|i} p_{i0} \text{ for all } i, j \quad \text{...(ii)}$$

and

$$P(x_i | y_j) = p_{j|i} (p_{i0}/p_{0j}) \text{ for all } i, j. \quad \text{...(iii)}$$

The relations (ii) give the joint probabilities of sending a symbol x_i and receiving the symbol y_j, while relations (iii) give the *backward* channel probabilities given that an output y_j has been received.

Illustration. *Consider a binary channel with input symbols* $A = \{0, 1\}$, *output symbols* $B = \{0, 1\}$ *and the channel matrix* $\begin{pmatrix} 2/3 & 1/3 \\ 1/10 & 9/10 \end{pmatrix}$

Let us assume the input probabilities $p_{10} = 3/4$, $p_{20} = 1/4$.

Using relation *(i)*, the following output probabilities are obtained :

$$p_{01} = \frac{3}{4} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{10} = \frac{21}{40}, p_{02} = \frac{3}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{9}{10} = \frac{19}{40}.$$

The conditional backward input probabilities are obtained by using (iii) :

$$P(0|0) = \frac{2/3 \,.\, 3/4}{21/40} = \frac{20}{21}, P(0|1) = \frac{1/3 \,.\, 3/4}{19/40} = \frac{10}{19},$$

$$P(1|0) = \frac{1/10 \,.\, 1/4}{21/40} = \frac{1}{21}, P(1|1) = \frac{9/10 \,.\, 1/4}{19/40} = \frac{9}{19}.$$

The joint probabilities are obtained by relation *(ii)* :

$$P(0, 0) = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}, \quad P(0, 1) = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4},$$

$$P(1, 0) = \frac{1}{10} \cdot \frac{1}{4} = \frac{1}{40}, P(1, 1) = \frac{9}{10} \cdot \frac{1}{4} = \frac{9}{40}.$$

26.5-4. Noiseless Channels

Definition. (Noiseless Channels). *A channel described by a channel matrix with one and only one non-zero element in each column is called a* ***Noiseless Channel.***

For example, (i) A *binary symmetric channel* with $p = 0$ or 1 is a *noiseless channel.*

(ii) The channel represented by the following channel matrix is a noiseless channel

$$\begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3/5 & 5/10 & 1/10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

26.6. A MEASURE OF INFORMATION

An amount of information is virtually a search for statistical parameter associated with a probability scheme. This parameter should indicate a relative meassage of uncertainty applicable to the occurrence of each particular meassage in the set of meassages.

Now, to obtain a formula for the amount of information suppose there are n distinct models

$$[m_1, m_2, m_3, \ldots, m_n]$$

of a particular machine. The problem is to select a machine from this list.

The desired amount of information $I(m_k)$ associated with the selection of a particular model m_k must be a function of the probability of selecting m_k, *i.e.*

$$I(m_k) = f(P\{m_k\}) \quad \text{...(26.2)}$$

Now assuming, for simplicity, that each of the model $m_1, m_2, \ldots, m_n$ is selected with an equal probability,

$$P\{m_1\} = P\{m_2\} = P\{m_3\} = \ldots = P\{m_n\} = 1/n \quad \text{...(26.3)}$$

Then equation (26.2) becomes

$$I_1(m_k) = f(1/n) \quad \text{...(26.4)}$$

which means that the amount of information is the function of n.

Further, assume that each piece of the machine can be ordered in one of m distinct colours. Again, for simplicity, if selection of colours is also assumed to have equal probabilities, then the amount of information associated with the selection of a colour c_j among $[c_1, c_2, c_3, \ldots, c_m]$ is

$$I_2(c_j) = f\{P(c_j)\} = f(1/m) \quad \text{...(26.5)}$$

where the function f must be the same as used in equation (26.4).

Finally, assume that the selection is done in two ways :

(i) First, select the machine and colour, the two selections being independent of each other. In this case

$$I(m_k \text{ and } c_j) = I_1(m_k) + I_2(c_j) \quad \text{or} \quad I(m_k \text{ and } c_j) = f(1/n) + f(1/m) \qquad ...(26.6)$$

(ii) Alternatively, select the machine and its colour simultaneously as one selection from mn possible number of selections with equal probability. Hence

$$I(m_k \text{ and } c_j) = f(1/mn) \qquad ...(26.7)$$

The equations (26.6) and (26.7) give

$$f(1/n) + f(1/m) = f(1/mn) \qquad ...(26.8)$$

which is a functional equation.

The functional equation has one of the solutions given by

$$f(x) = \log(1/x) \quad \text{or} \quad f(x) = -\log x \qquad ...(26.9)$$

Substitute in equation (26.8) the values

$$f(1/n) = \log n, \; f(1/m) = \log m, \text{ and } f(1/mn) = \log mn$$

to obtain the result

$$\log n + \log m = \log mn \qquad ...(26.10)$$

which is always true.

Example 3. *Suppose a baby has just been born at a neighbour's house and the question is asked, "whether the baby is a boy or a girl ?" The answer, "It is a boy" then gives a specific amount of information, according to eqn. (26.9), namely;*

$$\textit{Amount of information} = -\log \tfrac{1}{2} = \log 2.$$

[since it has been assumed that the baby was equally likely to have been a boy or a girl, so the probability of its being a boy is ½].

The numerical value of the amount of information in the above equation depends upon what logarithmic base is used. If 2 is used as the base of logarithms, then

$$\textit{Amount of information} = \log_2 2 = 1, \qquad ...(26.11)$$

which may be called a *binary digit.* In short, this unit is commonly known as a *bit.*

If 10 is used as the base of logarithm, a unit information may be called a *decimal digit.* In short, this unit is called as a *decit.*

Example 4. *Fig. 26.3 shows that a large field is divided into 64 squares. In the dark night, a cow has entered in this field. This cow is located by a member of searching party who sends back an information giving the location of the cow as the 43rd square. Calculate the amount of information obtained in the reception of this message.*

Before the message was received, the cow was just as likely to be in any one of the 64 squares as any other. Therefore, the probability that the cow was in 43rd square was $p = 1/64$.

For the certainty of the location of the cow, it is assumed that the information is completely reliable. The quantity of information received with the message is

$$-\log_2 p = -\log_2 \tfrac{1}{64} = \log_2 64 = 6 \log_2 2$$

Therefore, information = 6 bits. ...(26.12)

Alternatively, this message was received, the cow was in the square of the 6th row and 3rd column of the field. Again, one is interested to know the amount of information received with this meassage. Before the information was received, the cow was equally likely to have been in any of the different columns. Accordingly the probability that it was in column C_3 was

$$p_c = \tfrac{1}{3} \qquad ...(26.13)$$

Similarly, the probability that it was in the row R_6 was

$$p_R = \tfrac{1}{8}. \qquad ...(26.14)$$

If it is agreed, when the searching party is sent out, that a two symbol information will be sent in which the first symbol stands for the column and the second symbol stands for the row of the square in which the cow is located, then the message "$C_3 R_6$" locates the cow. In this information, the first symbol gives

$$\log(1/p_c) = \log 8 = 3 \text{ bits} \qquad ...(26.15)$$

of information, and the second symbol gives

$$\log(1/p_R) = \log 8 = 3 \text{ bits} \qquad \ldots(26.16)$$

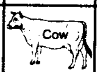
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
R_1	1	2	3	4	5	6	7	8
R_2	9	10	11	12	13	14	15	16
R_3	17	18	19	20	21	22	23	24
R_4	25	26	27	28	29	30	31	32
R_5	33	34	35	36	37	38	39	40
R_6	41	42	 43	44	45	46	47	48
R_7	49	50	51	52	53	54	55	56
R_8	57	58	59	60	61	62	63	64

Fig. 26.3. Numbering of squares of field into which cow has strayed

of information. The total amount of information necessary to locate the cow is thus 6 bits, in agreement with equation (26.12).

Q. Define information function. Describe its various requirements. Based on these requirements, characterise the information function.

26.7. ILLUSTRATIVE EXAMPLES

Example 5. *In a certain community, 25% of all girls are blondes and 75% of all blondes have blue eyes. Also 50% of all girls in the community have blue eyes. If you know that a girl has blue eyes, how much additional information do you being informed that she is blonde ?*

Solution. Let

$p_1 = p$ (blonde) = 0.25 = probability that a girl is blonde in absence of knowledge of the colour in her eyes.

$p_2 = p_{\text{blonde}}$ (blue eyes) = 0.75 = probability that a blonde has blue eyes.

$p_3 = p$ (blue eyes) = 0.50 = probability that a girl has blue eyes in the absence of knowledge of the colour of her hair.

$p_4 = p$ (blonde, blue eyes) = probability that a girl is blonde and has blue eyes.

$p_x = p_{\text{blue eyes}}$ (blonde) = ? = probability that a blue eyed girl is blonde and has blue eyes.

Then,
$$p_4 = p_1 p_2 = p_3 p_x \quad \text{or} \quad p_x = \frac{p_1 p_2}{p_3}$$

If a girl has blue eyes, the additional information obtained by being informed that she is blonde, is

$$\log(1/p_x) = \log(p_3/p_1p_2) = -\log p_1 - \log p_2 + \log p_3$$
$$= \log 4 - \log 4/3 - \log 2 = 1.42 \text{ bits.}$$ **Ans.**

Example 6. *A man is informed that, when a pair of dice were rolled, the result was a seven. How much information is there in this message.*

Solution. Do it yourself.

Example 7. *An alphabet consists of 8 consonants and 8 vowels. Suppose all letters of the alphabet are equally probable and there is no inter-symbol influence. If consonants are always understood correctly, but vowels are understood correctly only half of the time being mistaken for other vowels the other half of the time all vowels being involved in errors the same percentage of the time. What is the average rate of information transmission.*

Solution. According to conditions of the probelm, 50% of the letters received will be correct consonants, 25% will be correct vowels, and 25% incorrect vowels. Now first calculate the amount of information received in each of these cases.

Case I. Correct Consonants

Here p_A = probability before reception = 1/16, p_B = probability after reception = 1.

Therefore, information/letter $= \log\left(\frac{1}{1/16}\right) = \log 16 = 4$ ***bits/letter.***

Case II. Correct Vowels

Here p_A = probability before reception = 1/16, p_B = probability after reception = 1/2.

Therefore, information/letter $= \log\left(\frac{1/2}{1/16}\right) = \log 8 = 3$ bits/letter.

Case III. Incorrect Vowels

In this case, p_A = probability before reception = 1/16.

Now, find the probability after reception, suppose an E is sent and a U is received (Fig. 26.4). The problem is to find the probability that an E was sent as a consequence of receiving a U. Before reception, the proabability of E was 1/16, since 16 letters have equal probabilities. After receiving a U, a vowel was sent, and the probability of a U is 1/2. The other half of the probability is divided equally among other 7 vowels, one of which is E. Therefore, the probability that E was sent is $1/7 \times 1/2 = 1/14$.

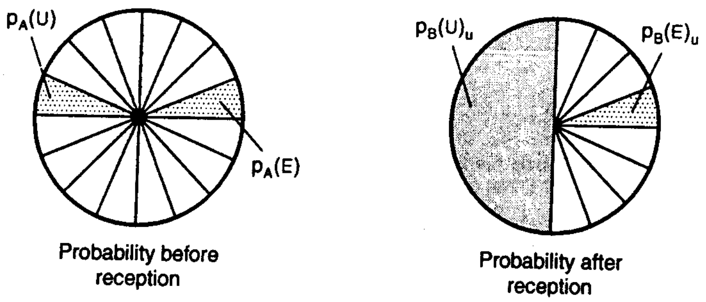


Fig. 26.4

Therefore, $p_B = p_B(E)_U$ = probability after reception = 1/14.

Thus, information/letter $= \log\left(\frac{1/14}{1/16}\right)$

$$= \log 16 - \log 14 = 4 - 3.8 = 0.2 \text{ bits/letter.}$$

It is interesting to note that the information received with an incorrect vowel is positive in this case. Since the reception of the incorrect vowel actually inceases the probability of transmission of the correct vowel.

Finally, for the average information per symbol, take 50% of case I and 25% each of case II and III. Therefore, average information/symbol

$$= 0.5 \times 4 + 0.25 \times 3 + 0.35 \times 0.2 = 2.8 \text{ bits/symbol.}$$ **Ans.**

26.8. A MEASURE OF UNCERTAINTY : ENTROPY

Let S be the sample space of events belonging to a random experiment. Compose this sample space into a finite number of mutually exclusive events $E_1, E_2, E_3, \ldots, E_n$, whose respective probabilities $p_1, p_2, p_3, \ldots, p_n$ are assumed to be known. These two sets can be represented in matrix form as

$$[E] = [E_1, E_2, E_3, \dots, E_n], \text{ where } \bigcup_{k=1}^{n} E_k = U \qquad \text{...(26.17)}$$

$$[P] = [p_1, p_2, p_3, \dots, p_n] \quad , \text{where } \sum_{k=1}^{n} p_k = 1 \qquad \text{...(26.18)}$$

Now, equations (26.17) and (26.18) contain all the information about the probability space which is called a *complete finite scheme.*

For example,
$$\begin{bmatrix} E \\ P \end{bmatrix} = \begin{bmatrix} E_1 & E_2 & E_3 & E_4 \\ 0.1 & 0.4 & 0.3 & 0.2 \end{bmatrix}$$

The fundamental problem of interest is to associate a measure of uncertainty, $H(p_1, p_2, \dots, p_n)$, with such probability schemes.

Shannon and Wiener have suggested the following measure of uncertainty (**entropy***) associated with the sample space of a complete finite scheme

$$H(X) = -\sum_{i=1}^{n} p_i \log p_i \qquad \text{...(26.19)}$$

The *average amount of information* or entropy of a finite complete probability scheme is defiend by

$$H(X) = I(\overline{E_k}) = -\sum_{k=1}^{n} p_k \log p_k, \qquad \text{...(26.20)}$$

where the random variable X is defined over the sample space of events S, and events satisfy the equations (26.17) and (26.18).

Validity of equaton (26.20) can be easily verified. If $-\log p_k$ denotes the measure of uncertainty associated with the event E_k, then the mean or the expected value of the uncertainty associated with probability scheme will be

$$\sum_{i=1}^{n} p_k (-\log p_k).$$

26.9. PROPERTIES OF ENTROPY FUNCTION (*H*)

The entropy function $H(p_1, p_2, \dots, p_n)$ possesses the following basic properties :

(1) *Continuity Property* : *$H(p_1, p_2 \dots\dots, p_n)$ is continuous in $p_k \; \forall \; 0 \le p_k \le 1$.*

Proof. The entropy function $H(p_1, p_2, \dots, p_n)$ is continuous for each and every independent variable p_k in the interval [0, 1]. To prove this,

$$-H(p_1, p_2, \dots, p_n) = p_1 \log p_1 + p_2 \log p_2 + \dots + p_n \log p_n$$
$$= p_1 \log p_1 + p_2 \log p_2 + \dots + p_{n-1} \log p_{n-1} + (1 - p_1 - p_2 - \dots - p_{n-1}) \log (1 - p_1 - p_2 - \dots - p_{n-1}) \qquad \text{...(26.21)}$$

It is seen that all independent variables $p_1, p_2, \dots, p_{n-1}$ and $(1 - p_1 - p_2 - \dots - p_{n-1})$ are continuous in (0,1) and that the logarithm of a continuous function is continuous itself.

(2) *Symmetric Property* : $p(p_k, 1 - p_k) = H(1 - p_k, p_k)$, where $k = 1, 2, \dots, n$.

Proof. The entropy function is obviously a symmetrical function in all variables.

(3) *External (Maximum) Property* : *Max.* $H(p_1, p_2, \dots, p_n) = H(1/n, 1/n, \dots, 1/n)$.

[Delhi (OR) 92]

Proof. In dynamic programming (*see solved example*), it has been proved that the entropy function

*Boltzmann, a famous physicist, used the symbol *H* for the expression on the right-hand side of eqn. (26.19) and showed that, in idealized system, it is proportional to the *thermodynamic* quantity entropy. It was also realized at that time that there was a close relation between entropy and information. Because of this historical background, it was natural that ***Shannon*** should have used the symbol *H* for this summation and should have called it *entropy.*

** All logarithms are to the base 2 unless otherwise stated.

$$H = -\Sigma\, p_i \log p_i \qquad ...(26.22)$$

has a maximum value when all individual probabilities are equal, *i.e.*

$$p_1 = p_2 = \ldots = p_n = 1/n \qquad ...(26.23)$$

Hence, max $H(p_1, p_2, p_3, \ldots, p_n) = H(1/n, 1/n, \ldots, 1/n)$.

(4) ***Additive Property*** :

$$H(p_1, p_2, \ldots, p_{n-1}, q_1, q_2, \ldots, q_m) = H(p_1, p_2, \ldots, p_{n-1}, p_n) + p_n H\left(\frac{q_1}{p_n}, \frac{q_2}{p_n}, \ldots, \frac{q_m}{p_n}\right)$$

where $$p_n = \sum_{k=1}^{m} q_k .$$

Proof. The validity of this property can be proved by reducing the left member to an identical form with the right member of above equation.

First, assume that the event E_n with probability p_n is divided into disjoint subsets $F_1, F_2, F_3, \ldots, F_m$ with respective probabilities $q_1, q_2, q_3, \ldots, q_m$. Thus,

$$p_n = q_1 + q_2 + q_3 + \ldots + q_m \text{ or } p_n = \sum_{k=1}^{m} q_k . \qquad ...(26.24)$$

Now, L.H.S. $= H(p_1, p_2, \ldots, p_{n-1}, q_1, q_2, q_3, \ldots, q_m)$

$$= -\sum_{k=1}^{n-1} p_k \log p_k - \sum_{k=1}^{m} q_k \log q_k \qquad \text{[from the equation (26.19)]}$$

$$= -\left[\sum_{k=1}^{n} p_k \log p_k - p_n \log p_n\right] - \sum_{k=1}^{m} q_k \log q_k$$

$$= H(p_1, p_2, \ldots, p_n) + \left[p_n \log p_n - \sum_{k=1}^{m} q_k \log q_k\right]$$

But, $$p_n \log p_n - \sum_{k=1}^{m} q_k \log q_k = p_n\left[\frac{p_n}{p_n} \log p_n\right] - p_n \sum_{k=1}^{m}\left[\frac{q_k}{p_n} \log q_k\right]$$

$$= p_n \sum_{k=1}^{m} \frac{q_k}{p_n} \log p_n - p_n \sum_{k=1}^{m} \frac{q_k}{p_n} \log q_k \quad \text{[using } p_n = \sum_{k=1} q_k \text{ from the equation (26.24)]}$$

$$= -p_n \sum_{k=1}^{m} \frac{q_k}{p_n} (\log q_k - \log p_n) \qquad = -p_n \sum_{k=1}^{m} \frac{q_k}{p_n} \log \frac{q_k}{p_n} .$$

Therefore, L.H.S. $$= H(p_1, p_2, \ldots, p_n) + \left[-p_n \sum_{k=1}^{m} \frac{q_k}{p_n} \log \frac{q_k}{p_n}\right]$$

$$= H(p_1, p_2, \ldots, p_n) + p_n H\left(\frac{q_1}{p_n}, \frac{q_2}{p_n}, \frac{q_3}{p_n}, \ldots, \frac{q_m}{p_n}\right) = \text{R.H.S.}$$

This proves the additive property of the H-function.

Note. It should be noted that partitioning of events into subevents cannot decrease the entropy of the system, *i.e.*

$$H(p_1, p_2, \ldots, p_{n-1}, q_1, q_2, \ldots, q_m) \geq H(p_1, p_2, \ldots, p_{n-1}, p_n) \qquad ...(26.25)$$

Moreover, the following theorem establishes an important property of the entropy function.

Theorem 26.1. *Let* $p_1, p_2, \ldots, p_m$ *and* $q_1, q_2, \ldots, q_m$ *be arbitrary non-negative numbers with* $\sum_{i=1}^{m} p_i = \sum_{i=1}^{m} q_i$. *Then* $-\sum_{i=1}^{m} p_i \log p_i \leq -\sum_{i=1}^{m} p_i \log q_i$ *with equality, if and only if* $p_i = q_i$ *for all i.*

Proof. Since the logarithm is a convex function, we have the inequality $\log x \leq x - 1$ with equality iff $x = 1$ as shown in the following figure.

If we take $x = q_i/p_i$, then

$$\log (q_i/p_i) \leq (q_i/p_i) - 1. \quad ...(i)$$

with equality, if and only if, $q_i = p_i$.

Multiplying *(i)* by p_i and summing over i, we get

$$\sum_{i=1}^{m} p_i \log (q_i/p_i) \leq \sum_{i=1}^{m} (q_i - p_i) = 1 - 1 = 0$$

with equality, if and only if, $q_i = p_i$ for all i,

This proves that

$$\sum_{i=1}^{m} p_i \log q_i \leq \sum_{i=1}^{m} p_i \log p_i$$

or

$$- \sum_{i=1}^{m} p_i \log p_i \leq - \sum_{i=1}^{m} p_i \log q_i$$

with equality, if and only if, $p_i = q_i$ for all i.

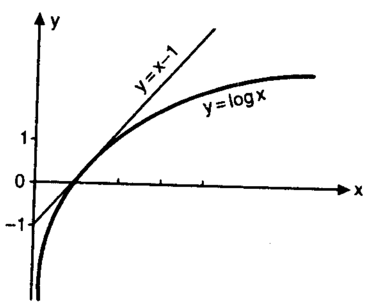


Fig. 26.5. The Natural Logarithm of *x*.

Illustrative Examples

Example 8. *Evaluate the average uncertainty associated with the probability space of events shown in Fig. 26.6.*

$$\begin{bmatrix} \text{Events} \\ \text{Probability} \end{bmatrix} = \begin{bmatrix} A & B & C & D \\ 1/2 & 1/4 & 1/8 & 1/8 \end{bmatrix}$$

Solution. Here $p_1 = 1/2$, $p_2 = 1/4$, $p_3 = 1/8$, $p_4 = 1/8$.

Substituting these values in equation (26.19)

$$H(1/2, 1/4, 1/8, 1/8) = -1/2 \log 1/2 - 1/4 \log 1/4 - 1/8 \log 1/8 - 1/8 \log 1/8$$

$$= 1/2 \log 2 + 1/4 \log 4 + 1/8 \log 8 + 1/8 \log 8 = 1/8\,[4 \log 2 + 2 \log 2^2 + 2 \log 2^3]$$

$$= 1/8\,[4 + 4 + 6] \log 2 = 14/8 \text{ bits} \qquad (\text{since } \log_2 2 = 1)$$

Ans.

Example 9. *Verify the rule of the additivity of entropies for events A, B and C with probabilities* 1/5, 4/15 *and* 8/15, *respectively.*

Solution. The entropy function H is defined as

$$H(p_1, p_2, \ldots\ldots, p_n) = - \sum_{i=1}^{n} p_i \log p_i$$

where p_i *'s* are the probabilities associated with the given probability distribution.

Here $p_1 = 1/5$, $p_2 = 4/15$ and $p_3 = 8/15$ Therefore,

$$H(1/5, 4/15, 8/15) = -\frac{1}{5} \log\left(\frac{1}{5}\right) - \frac{4}{15} \log\left(\frac{4}{15}\right) - \frac{8}{15} \log\left(\frac{8}{15}\right)$$

$$= \frac{1}{5} \log 5 + \frac{4}{15} \log \frac{15}{4} + \frac{8}{15} \log \frac{15}{8}$$

$$= \frac{1}{15}\,[15 \log 5 + 12 \log 3 - 32] \qquad (\because \log_2 2 = 1)$$

$$= \log 5 + \frac{4}{5} \log 3 - \frac{32}{15}$$

Note. It is easy to verify that $H(1/5, 4/15, 8/15) = H(1/5, 4/5) + 4/5\, H(1/3, 2/3)$.

26.10. JOINT AND CONDITIONAL ENTROPIES

26.10–1. Relation Between Joint and Marginal Entropies

Let us consider two sets of messages : $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$,
where x_i's are the messages sent *(channel input)* and y_i's are the messages received *(channel output)*.

Let $p_{ij} = P(X = x_i, Y = y_j)$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ denote the probability of the joint event that message x_i is sent and message y_j is received. Now we are able to study the relationship between *joint,* *conditional* and *marginal (individual)*, informations associated with the bivariate probability distribution (p_{ij}).

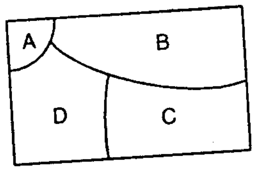


Fig. 26.6

We may define the marginal probability distributions of X and Y by

$$p_{i0} = \sum_{j=1}^{n} p_{ij} \text{ and } p_{0j} = \sum_{i=1}^{m} p_{ij} \text{ for all } i, j.$$

Then, obviously the *marginal entropies* of the two marginal distributions are given by

$$H(X) = -\sum_{i=1}^{m} p_{i0} \log p_{i0} \quad \text{and} \quad H(Y) = -\sum_{j=1}^{n} p_{0j} \log p_{0j}.$$

The entropy $H(X)$ measures the uncertainty of the message sent (irrespective of the message received) and $H(Y)$ performs the same role for the message received.

The *joint entropy* is the entropy of the joint distribution of the messages sent and received, and is therefore given by

$$H(X, Y) = -\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \log p_{ij}.$$

It may be observed that :

$$\text{Max } H(X, Y) = \log mn = \log m + \log n = \max H(X) + \max H(Y).$$

Theorem 26.2. *$H(X, Y) \le H(X) + H(Y)$ with equality, if and only if, X and Y are independent.*

Proof. We may write

$$H(X) + H(Y) = -\sum_{i=1}^{m} p_{i0} \log p_{i0} - \sum_{j=1}^{n} p_{0j} \log p_{0j}$$

$$= -\sum_{i=1}^{m} \left(\sum_{j=1}^{n} p_{ij} \right) \log p_{i0} - \sum_{j=1}^{n} \left(\sum_{i=1}^{m} p_{ij} \right) \log p_{0j}$$

$$= -\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \log (p_{i0}\, p_{0j}) = -\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \log q_{ij}, \text{ where } q_{ij} = p_{i0}\, p_{0j}. \quad \text{...(i)}$$

Also, by definition

$$H(X, Y) = -\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \log p_{ij}. \quad \text{...(ii)}$$

But, we observe that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} q_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{i0}\, p_{0j} = \left(\sum_{i=1}^{m} p_{i0} \right) \left(\sum_{j=1}^{n} p_{0j} \right) = 1 = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}$$

By virtue of *Theorem 26.1, (i)* and *(ii)*, it follows that

$$H(X, Y) \le H(X) + H(Y)$$

with equality, if and only if, $q_{ij} = p_{ij}$ for all i and j. The condition for equality reduces to $p_{i0}\, p_{0j} = p_{ij}$ meaning thereby X and Y are independent.

Q. 1. Show that $H(X)$ achieves its maximum if all the valuos of X are equi-probable.

2. Let X and Y be two discrete radom variables, each taking a finite number of values. Show that $H(Y \mid X) \le H(Y)$ with equality iff X and Y are independent **[Delhi (OR) 92]**

26.10–2. Conditional Entropies

Consider two finite discrete sample spaces S_1 and S_2 and their product space S. In S_1 and S_2, select complete sets of events in the sense of equations (26.17) and (26.18).

$$\{E\} = [E_1, E_2, \ldots, E_n] \quad \text{...(26.26)}$$

$$\{F\} = [F_1, F_2, \ldots, F_m] \quad \text{...(26.27)}$$

$$\begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.25 & 0.00 & 0.00 & 0.00 \\ x_2 & 0.10 & 0.30 & 0.00 & 0.00 \\ x_3 & 0.00 & 0.05 & 0.10 & 0.00 \\ x_4 & 0.00 & 0.00 & 0.05 & 0.10 \\ x_5 & 0.00 & 0.00 & 0.05 & 0.00 \end{array}$$

Determine the Marginal, Conditional and Joint entropies for this channel (Assume 0 log 0 ≅ 0).

Solution. The channel is described here by the joint probabilities p_{ij}, $i = 1, 2, \ldots, 5$ and $j = 1, 2, \ldots, 4$. The conditional and marginal probabilities are easily obtained from p_{ij}'s as given below :

$p_{10} = 0.25 + 0.00 = 0.25$, $p_{20} = 0.10 + 0.30 = 0.40$, $p_{30} = 0.05 + 0.10 = 0.15$,
$p_{40} = 0.05 + 0.10 = 0.15$, $p_{50} = 0.05 + 0.00 = 0.05$.

Similarly, $p_{01} = 0.35$, $p_{02} = 0.35$, $p_{03} = 0.20$, $p_{04} = 0.10$.

By using the result $(p_{j|i} = p_{ij}/p_{i0})$, the conditional probabilities are given in the following channel matrix :

Conditional Prob. Matrix $(p_{j|i})$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 1/4 & 3/4 & 0 & 0 \\ 3 & 0 & 1/3 & 2/3 & 0 \\ 4 & 0 & 0 & 1/3 & 2/3 \\ 5 & 0 & 0 & 1 & 0 \end{array}$$

Now various entropies associated with this channel are obtained (taking all logarithms to the base 2).

Marginal Entropies :

$$H(X) = -\sum_{i=1}^{5} p_{i0} \log p_{i0}$$

$$= -[(.25)\log(.25) + (.40)\log(.40) + 2\,(.15)\log(.15) + (.05)\log(.05)]$$

$$= \frac{1}{4}\log 4 + \frac{2}{5}\log\frac{5}{2} + \frac{3}{10}\log\frac{20}{3} + \frac{1}{20}\log 20 = 1.3260 \text{ bits.}$$

$$H(Y) = -\sum_{j=1}^{4} p_{0j}\log p_{0j} = -[2\,(.35)\log(.35) + (.20)\log(.20) + (.10)\log(.10)]$$

$$= \frac{7}{10}\log(2.857) + \frac{1}{5}\log 5 + \frac{1}{10}\log 10$$

$$= 1.8556 \text{ bits}$$

Conditional Entropies :

$$H(Y \mid X) = -\sum_{i=1}^{5}\sum_{j=1}^{4} p_{ij}\log p_{j|i}$$

$$= (.25)\log 1 + (.10)\log 4 + (.30)\log(4/3) + (.05)\log 3$$
$$+ (.10)\log(3/2) + (.05)\log 3 + (.10)\log(3/2) + (.05)\log 1$$

$$= \frac{1}{10}\log 4 + \frac{3}{10}\log 4 - \frac{3}{10}\log 3 + \frac{1}{10}\log 3 + \frac{1}{5}\log 3 - \frac{1}{5}\log 2$$

$$= 1/10\,[4\log 4 - 2\log 2] = 6/10 = 0.60 \text{ bits.}$$

By *Theorem* 26.3, we have

$$H(X \mid Y) = H(X) + H(Y \mid X) - H(Y) = 1.3260 + 0.6000 - 1.8556 = 0.0704 \text{ bits.}$$

Joint Entropies :

$$H(X, Y) = H(X) + H(Y \mid X) = 1.3260 + 0.6000 = 1.9260 \text{ bits.}$$

26.12. SET OF AXIOMS FOR AN ENTROPY FUNCTION

Assume the following four conditions as axioms :

(1) Given a finite complete probability scheme $(p_1, p_2, .., p_n)$, then

$$\max H(p_1, p_2, p_3, \ldots, p_n) = H(1/n, \ldots, 1/n)$$

That is, the entropy function $H(p_1, p_2, \ldots, p_n)$ must take its maximum value when all events have equal probabilities $(p_1 = p_2 = \ldots = p_n = 1/n)$.

(2) For a joint finite complete scheme, associated entropies should satisfy the identity

$$H(X, Y) = H(X) + H(Y \mid X)$$

That is, the average information conveyed by (X, Y) is the sum of the average information given by X and that provided by Y when X is given.

(3) If an impossible event is added to a scheme, the entropy of the scheme should not be effected. Symbolically, $H(p_1, p_2, \ldots, p_n, 0) = H(p_1, p_2, \ldots, p_n)$.

(4) The entropy function is continuous with respect to all its arguments.

These axioms essentially lead to a unique expression for entropy of finite scheme. Therefore, uniqueness theorem is proved in the following section.

Q. 1. Discuss the application of maximum entropy principle in operations research **[Delhi (OR) 92]**

2. State and explain the fundamental theorem of information theory. **[Delhi (OR) 92]**

3. Discuss the maximum liklihood decision scheme in coding theory.

26.13. UNIQUENESS THEOREM

Theorem 26.4. *The only function which satisfies four axioms (in sec.* **26.13***) is*

$$H(p_1, p_2, \ldots, p_n) = \lambda \sum_{i=1}^{n} p_i \log p_i$$

where λ is an arbitrary positive number and the logarithm base is any number greater than 1.

Proof. Consider

$$H(1/n, 1/n, \ldots, 1/n) = f(n). \qquad \ldots(26.47)$$

Step 1. ***To show that $f(n) = \lambda \log n$.***

Since,

$$f(n) = H(1/n, 1/n, \ldots, 1/n) \le H\{1/(n+1), 1/(n+1), \ldots, 1/(n+1)\} = f(n+1) \qquad \ldots(26.48)$$

This inequality shows that $f(n)$ is a *non-decreasing* function of n.

Now, according to the *axiom* (2), for any complete probability scheme consisting of the sum of *m mutually* exclusive schemes,

$$H(X_1, X_2, \ldots, X_m) = H(X_1) + \ldots + H(X_m) = \sum_{k=1}^{m} H(X_k) \qquad \ldots(26.49)$$

If each scheme consists of r equally likely events, then

$$H(X_1, X_2, \ldots, X_m) = mf(r) = f(r^m)$$

where m and r being any arbitrary integers.

Now, take two integers t and n such that

$$r^m < t^n < r^{m+1} \quad \text{or} \quad m \log r < n \log t < (m+1) \log r \quad \text{or} \quad \frac{m}{n} < \frac{\log t}{\log r} < \frac{m+1}{n}. \qquad \ldots(26.50)$$

Since $f(n)$ is the non-decreasing function, so

$$f(r^m) \le f(t^n) \le f(r^{m+1}) \quad \text{or} \quad mf(r) \le nf(t) \le (m+1) f(r) \quad \text{or} \quad \frac{m}{n} \le \frac{f(t)}{f(r)} \le \frac{m+1}{n}. \qquad \ldots(26.51)$$

Now, comparing equations (26.50) and (26.51),

$$\left| \frac{f(t)}{f(r)} - \frac{\log t}{\log r} \right| \le \frac{1}{n}. \qquad \ldots(26.52)$$

As $n \to \infty$, for any positive integers r and t,

$$\frac{f(t)}{f(r)} = \frac{\log t}{\log r} = 0 \text{ or } \frac{f(t)}{\log t} = \frac{f(r)}{\log r} = \lambda \text{ (say) or}$$

$$f(t) = \lambda \log t, f(r) = \lambda \log r \text{ (say) or } f(n) = \lambda \log n. \quad ...(26.53)$$

Here, λ must be a positive constant because $f(t)$ is a non-decreasing function.

This proves the uniqueness theorem for the particular case when all events have equal probabilities.

Step 2. ***To prove uniqueness theorem when all probabilities are rational numbers (but not necessarily all equal).***

Let α be a common denominator for the different rational p_k,

and
$$p_k = \frac{\alpha_k}{\alpha}, \sum_{k=1}^{n} \alpha_k = \alpha, \alpha_k > \alpha; k = 1, 2, 3, \dots, n \quad ...(26.54a)$$

Now, transfer the problem to the case discussed in *Step* 1. To do so, consider a probability scheme X depending on X. Let the scheme Z consists of α equally likely events $[z_1, z_2, z_3, \dots, z_\alpha]$.

For simplicity, decompose these events into groups containing events $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ and α_n, respectively. Denote the decomposed scheme by Y. When the event X_k with the probability p_k occurs, then all events partitioned in the kth group occur with equal probability in scheme Y. Thus

$$H(1/\alpha_k, 1/\alpha_k, \dots, 1/\alpha_k) = \lambda \log \alpha_k = \lambda (\log p_k + \log \alpha) \quad ...(26.54b)$$

$$H(Y \mid X) = \sum_{k=1}^{n} p_k H\left(\frac{1}{\alpha_k}, \frac{1}{\alpha_k}, \dots, \frac{1}{\alpha_k}\right) = \lambda \sum_{k=1}^{n} p_k \log p_k + \lambda \log \alpha.$$

The totality of events in Z forms the sum of two schemes

$$H(Z) = H(X, Y) = f(\alpha) = \lambda \log \alpha.$$

But,
$$H(X, Y) = H(X) + H(Y \mid X) \text{ or } H(X) = H(X, Y) - H(Y \mid X)$$

$$= (\lambda \log \alpha) - [\lambda \sum_{k=1}^{n} p_k \log p_k + \lambda \log \alpha] = -\lambda \sum_{k=1}^{n} p_k \log p_k$$

Thus, the uniqueness theorem also holds when $p_1, p_2, \dots, p_n$ are rational numbers.

Step 3. Finally, the continuity axiom of the entropy function ensures that the *uniqueness theorem* is valid when $p_1, p_2, p_3, \dots, p_n$ are incommensurable.

This completes the proof of the theorem.

Q. Describe applications of maximum entropy principle in (i) Transportation Problem (ii) Production Scheduling.

26.14. TRANS-INFORMATION (MUTUAL INFORMATION)

Expected Mutual Information. Consider again the set of messages sent $X = \{x_1, x_2, \dots, x_m\}$ and the set of messages received $Y = \{y_1, y_2, \dots, y_n\}$. Then, the quantity

$$\underline{h}(x_i, y_j) = \log\left(\frac{p_{ij}}{p_{i0}\, p_{0j}}\right), i = 1, 2, \dots, m; j = 1, 2, \dots, n.$$

is known as the mutual information of the message sent x_i and the message received y_j. The following observations on $\underline{h}(x_i, y_j)$ are obvious :

(i) $\underline{h}(x_i, y_j) = 0$ whenever X and Y are independent.

(ii) $\underline{h}(x_i, y_j) >$ or < 0 according as $p_{j \mid i} >$ or $< p_{0j}$ for a fixed x_i.

Now, averaging all mn mutual information values with p_{ij}'s as weights, we obtain the *Expected Mutual Information* of X and Y, as

$$I(X, Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \log\left(\frac{p_{ij}}{p_{i0}\, p_{0j}}\right)$$

Theorem 26.5. $I(X; Y) = H(X) - H(X \mid Y) = H(Y) - H(Y \mid X)$

Proof. We have

$$H(X) - H(X \mid Y) = -\sum_{i=1}^{m} p_{i0} \log p_{i0} + \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij} \log p_{i \mid j}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij} \log\left(\frac{p_{i\mid j}}{p_{i0}}\right) \qquad \left[\text{using } p_{i0} = \sum_{j=1}^{n} p_{ij}\right]$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij} \log\left(\frac{p_{ij}}{p_{i0}\,p_{0j}}\right) \qquad [\text{using } p_{ij} = p_{i\mid j}\, p_{0j}]$$

$$= I(X ; Y).$$

Similarly, other equality relation holds.

Physical Interpretation. The expression $H(X) - H(X \mid Y)$ may be considered as the *reduction in uncertainty* about X and Y is revealed. Thus $I(X, Y)$ may also be called the amount of information conveyed by Y about X. Symmetry of $I(X, Y)$ then states that—

'The information conveyed about X by Y is the same as the information conveyed about Y by X'.

Also, $I(X, Y) = 0$ when X and Y are independent, *i.e.* no information is conveyed by a variable about another which is stochastically independent of the former.

Theorem 26.6. $I(X ; Y) = H(X) + H(Y) - H(X, Y)$.

Proof. Immediately follows from *Theorems* 26.5 & 26.3.

26.15. CHANNEL CAPACITY, EFFICIENCY AND REDUNDANCY

Channel Capacity. The trans-information $I(X ; Y)$ indicates a measure of the average information per symbol transmitted in the system. According to ***Shannon,*** in a discrete communication system, the *channel capacity* is the maximum of trans-information

$$C = \max I(X ; Y) = \max [H(X) - H(X \mid Y)] \qquad \text{...(26.56)}$$

For discrete noiseless channels, the channel capacity can be evaluated as follows :

Let $X = \{x_i\}$ be the alphabet of a source containing n symbols. Also, for a noise-free channel,

$$I(X ; Y) = H(X) = H(Y) = H(X, Y).$$

Thus,
$$C = \max I(X ; Y) = \max [H(X)] = \max \left[- \sum_{i=1}^{n} p\{x_i\} \log p\{x_i\}\right] \qquad \text{...(26.57)}$$

Since the maximum of $H(X)$ occurs when all symbols have equal probabilities, the channel capacity becomes

$$C = -\log(1/n) = \log n \text{ bits/symbol} \qquad \text{...(26.58)}$$

The capacity of a channel, as well as the rate of transmission of information through the channel can be equivalently expressed in bits/second (instead of bits/symbol). If symbols have a common duration of t sec., then channel capacity C per/sec. is given by

$$C_t = C/t \text{ bits/sec.} = (\log n)/t \text{ bits/sec.} \qquad \text{...(26.59)}$$

Redundancy. The difference between the actual rate of transmission of information $I(X ; Y)$ and its maximum possible value is defined as the *(absolute) redundancy* of the communication system. The ratio of absolute redundancy to the channel capacity is defined as the *relative redundancy* of the channel capacity.

Absolute redundancy for noise-free channel $= C - I(X ; Y)$ or $\log n - H(X)$. ...(28.60)

Relative redundancy for noise-free channel $= \dfrac{\log n - H(X)}{\log n}$ or $1 - \dfrac{H(X)}{\log n}$. ...(26.61)

Efficiency. The efficiency of the system may be defined as follows :

Efficiency of the noise-free channel $= \dfrac{I(X ; Y)}{\log n} = \dfrac{H(X)}{\log n} = 1 -$ relative redundancy ...(26.62)

Q. What do you understand by channel capacity and redundancy, efficiency ? Explain with an illustration.

Example 11. *Find the capacity of the memoryless channel specified by*

$$\begin{bmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

Solution. The capacity of the memoryless channel is given by

$$C = \max I(X\,;\,Y) = \max\,[H(X) + H(Y) - H(X\,,\,Y)]$$

The given channel is specified by

$$\begin{bmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

Therefore, the required capacity is given by

$$C = -\sum_{i=1}^{4} p_{i1} \log p_{i1} - \sum_{i=1}^{4} p_{i2} \log p_{i2} - \sum_{i=1}^{4} p_{i3} \log p_{i3} - \sum_{i=1}^{4} p_{i4} \log p_{i4}$$

where $p_{i1} = [1/2\,,\,1/4\,,\,1/4\,,\,0], p_{i2} = [1/4\,,\,1/4\,,\,1/4,\,1/4], p_{i3} = [0,\,0,\,1,\,0], p_{i4} = [1/2\,,\,0,\,0,\,1/2]$ for $i = 1, 2, 3, 4$.

Hence, $C = -\,[1/2 \log 1/2 + 2\,(1/4 \log 1/4)] + 4\,[(1/4\, Log\, 1/4) + 1 \log 1 + 2\,(1/2 \log 1/2)]$

$= [3/2 \log 2 + 3 \log 2] \quad = (3/2 + 3) \log 2 = 9/2$ bits/symbol.

26.16. ENCODING

First application of the notion of uncertainty discussed so far will be to the problem of efficient coding of messages to be sent over a 'noiseless' channel, *i.e.* a channel allowing perfect transmission from the source to the destination. In calculating the long run efficiency of communication system, the *average* length of a code word is of considerable interest. It is the quantity which is chosen to minimize.

Following are the elements of the noiseless coding problem :

(1) A *random variable X*, taking values $m_1\,,\,m_2\,,\,\ldots\,,\,m_N$ with prescribed probabilities $p\,\{m_1\}\,,\,p\,\{m_2\}\,,\,\ldots\,,\,p\,(m_N)$ respectively. X is to be observed independently over and over again, thus generating a sequence whose components belong to the set $\{m_1\,,\,m_2\,,\,\ldots\,,\,m_N\}$. Such a sequence is called a *massage.*

(2) A set $\{a_1\,,\,a_2\,,\,\ldots\,,\,a_D\}$ is called the set of *code character* or *the code alphabet;* each symbol m_i is to be assigned a finite sequence of code characters called the *code word* associated with m_i. For example, m_1 corresponds to $a_1\,,\,a_2$ and m_2 corresponds to $a_2\,,\,a_7\,,\,a_3\,,\,a_8$. The collection of all code words is called a *code.* Code words are assumed to be distinct.

(3) The objective of noiseless coding is to minimize the average code word length. Number of letters in a word is called the *length of the word.* If the code word associated with m_i is of length $n_i\,,\,i = 1, 2, \ldots\,,\,N$, then the problem is to determine codes that minimize the average length of messages.

$$L = \sum_{i=1}^{N} p\,\{m_i\}\,n_i\,,\,i = 1, 2, \ldots, N.$$

26.17. THE PROBLEM OF UNIQUE DECIPHERABILITY

Consider the following binary code

m_1	0
m_2	0 1 0
m_3	0 1
m_4	1 0